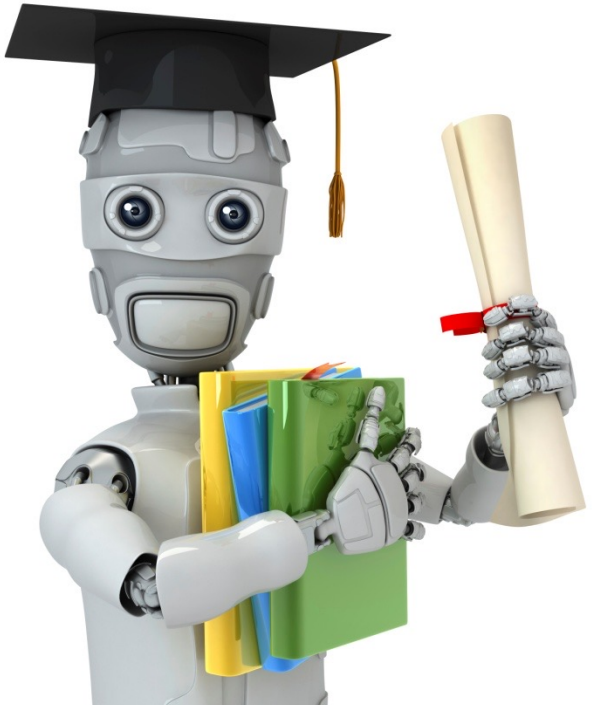


# Regression and Gradient Descent

Source: Intro. to Machine Learning
By Andrew Ng, Stanford, Coursera

**Training set of housing prices (Portland, OR)**

| Size in feet² (x) | Price ($) in 1000's (y) |
|---|---|
| 2104 | 460 |
| 1416 | 232 |
| 1534 | 315 |
| 852 | 178 |
| … | … |

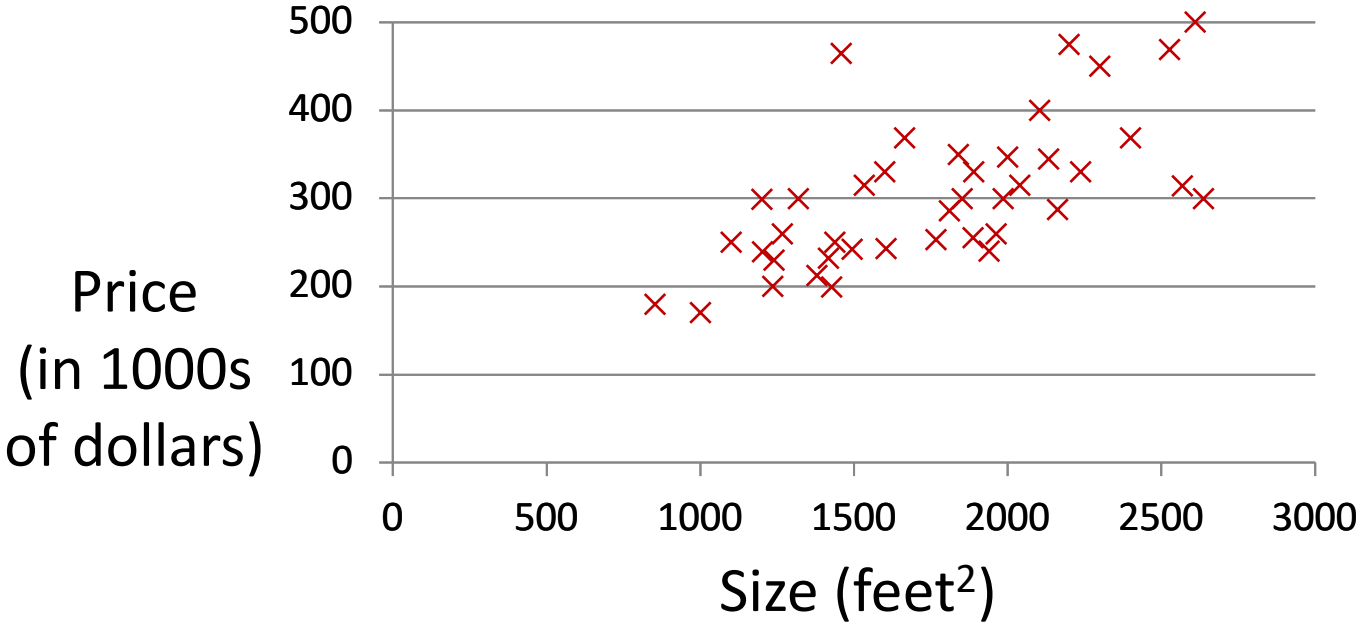
Notation:

**m** = Number of training examples

**x**'s = "input" variable / features

**y**'s = "output" variable / "target" variable

# Housing Prices (Portland, OR)




Regression Problem

Predict real-valued output

# Training Set

| Size in feet² (x) | Price ($) in 1000's (y) |
|---|---|
| 2104 | 460 |
| 1416 | 232 |
| 1534 | 315 |
| 852 | 178 |
| … | … |

Hypothesis: $h_\theta(x) = \theta_0 + \theta_1 x$

$\theta_i$'s: Parameters

How to choose $\theta_i$'s ?

Hypothesis:

$$h_\theta(x) = \theta_0 + \theta_1 x$$

Parameters:

$$\theta_0, \theta_1$$

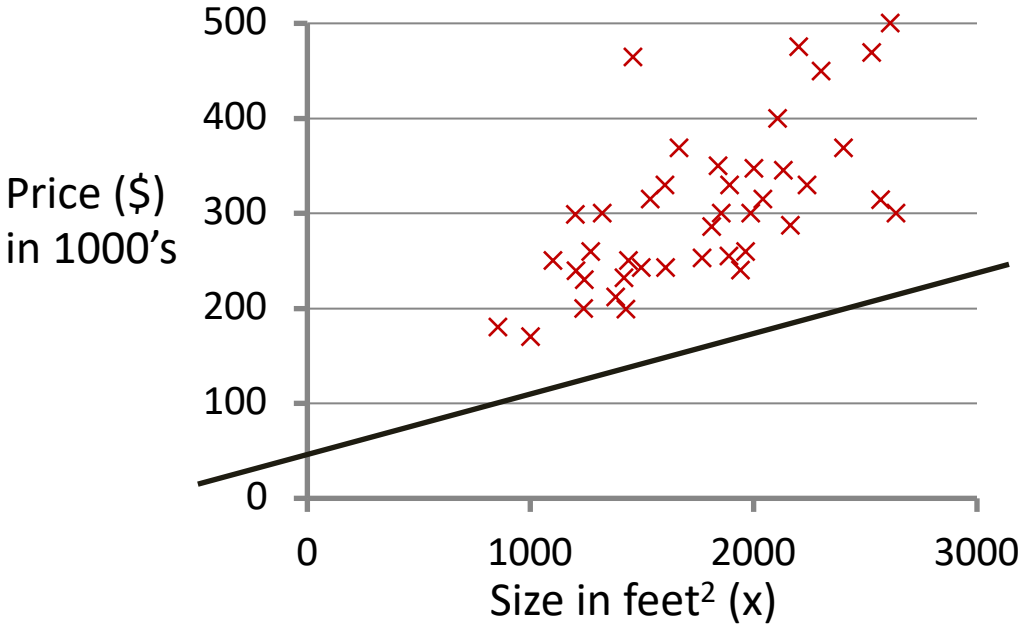
Cost Function (aka Loss Function):

$$J(\theta_0, \theta_1) = \frac{1}{2m} \sum_{i=1}^{m} \left( h_\theta(x^{(i)}) - y^{(i)} \right)^2$$

Goal: $\underset{\theta_0, \theta_1}{\text{minimize}}\ J(\theta_0, \theta_1)$

$h_\theta(x)$

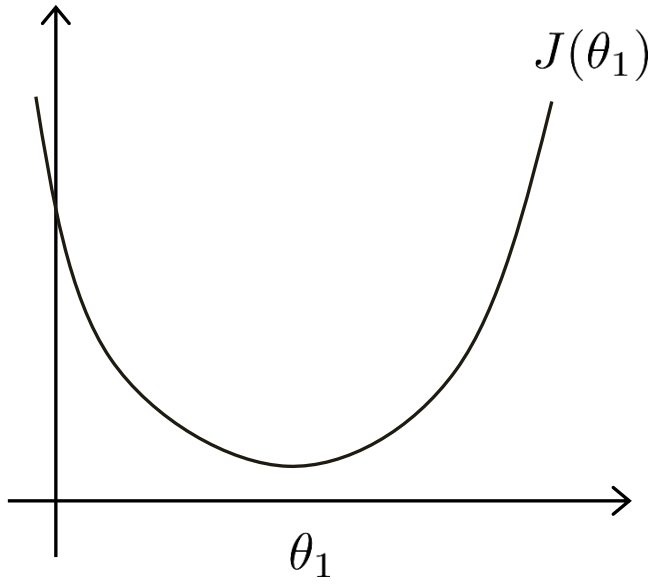
(for fixed $\theta_0, \theta_1$, this is a function of x)

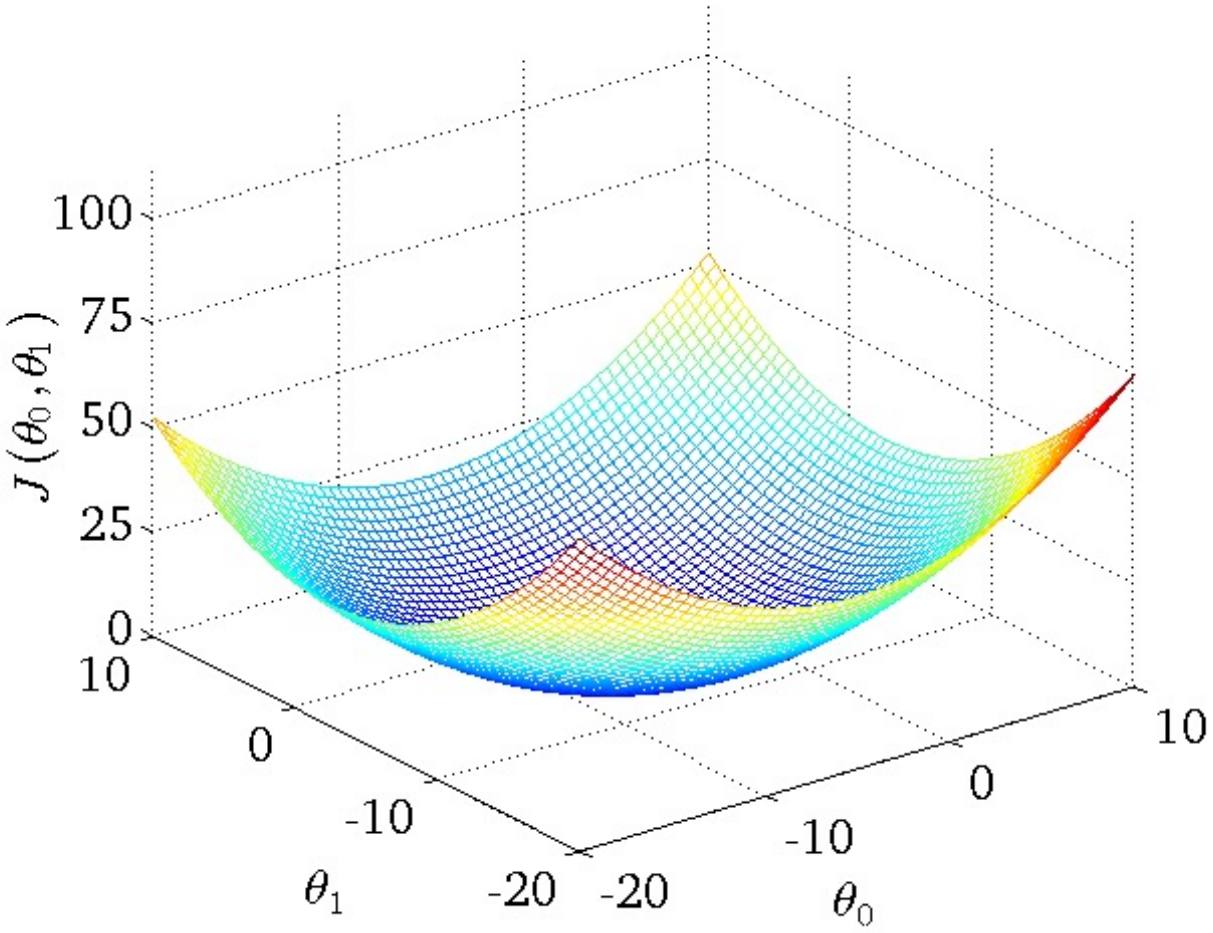



$$h_\theta(x) = 50 + 0.06x$$

$J(\theta_0, \theta_1)$

(function of the parameters $\theta_0, \theta_1$)




100
75
50
25
0
$J(\theta_0, \theta_1)$
10
0
-10
-20
$\theta_1$
-20
-10
0
10
$\theta_0$

# Gradient descent

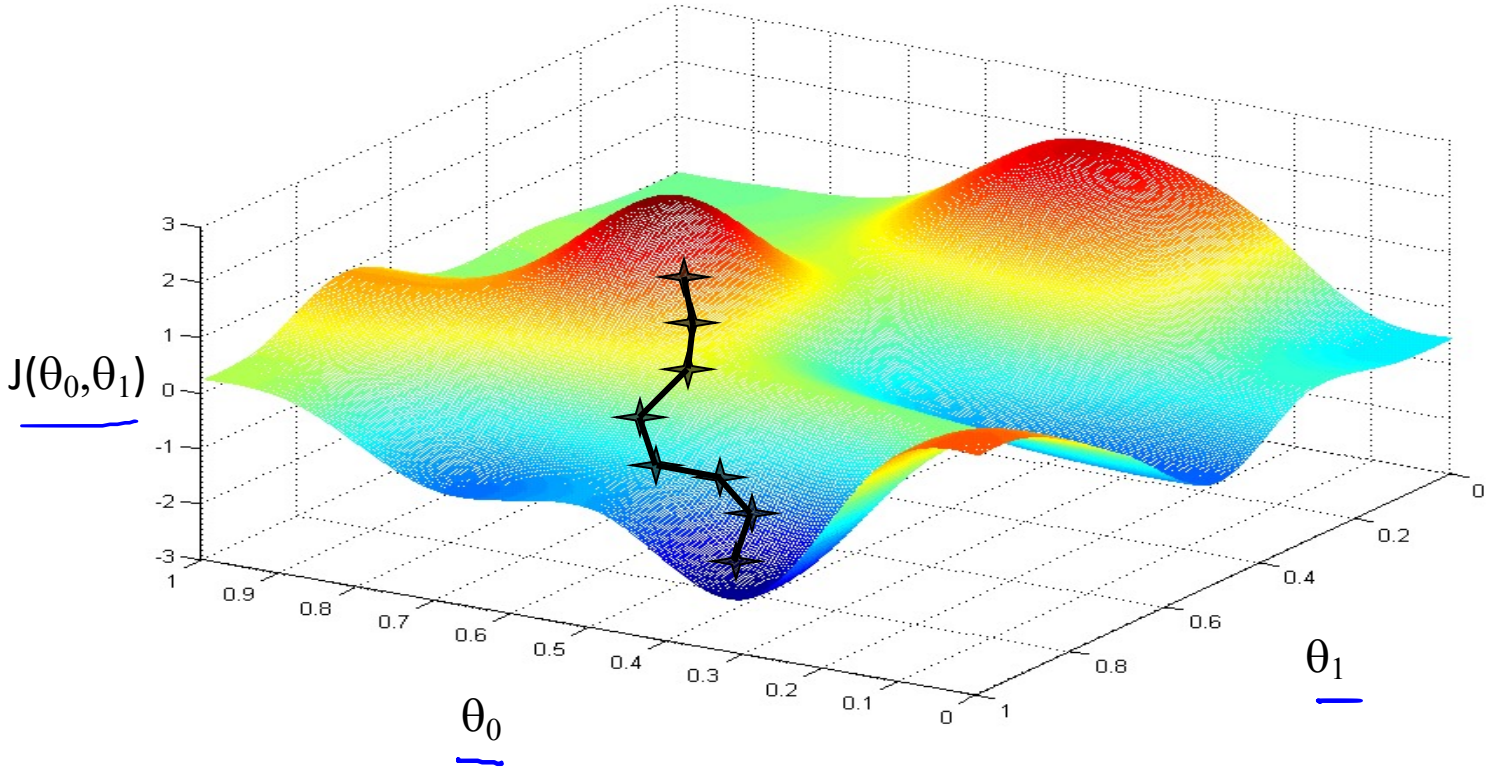
Have some function $J(\theta_0, \theta_1)$

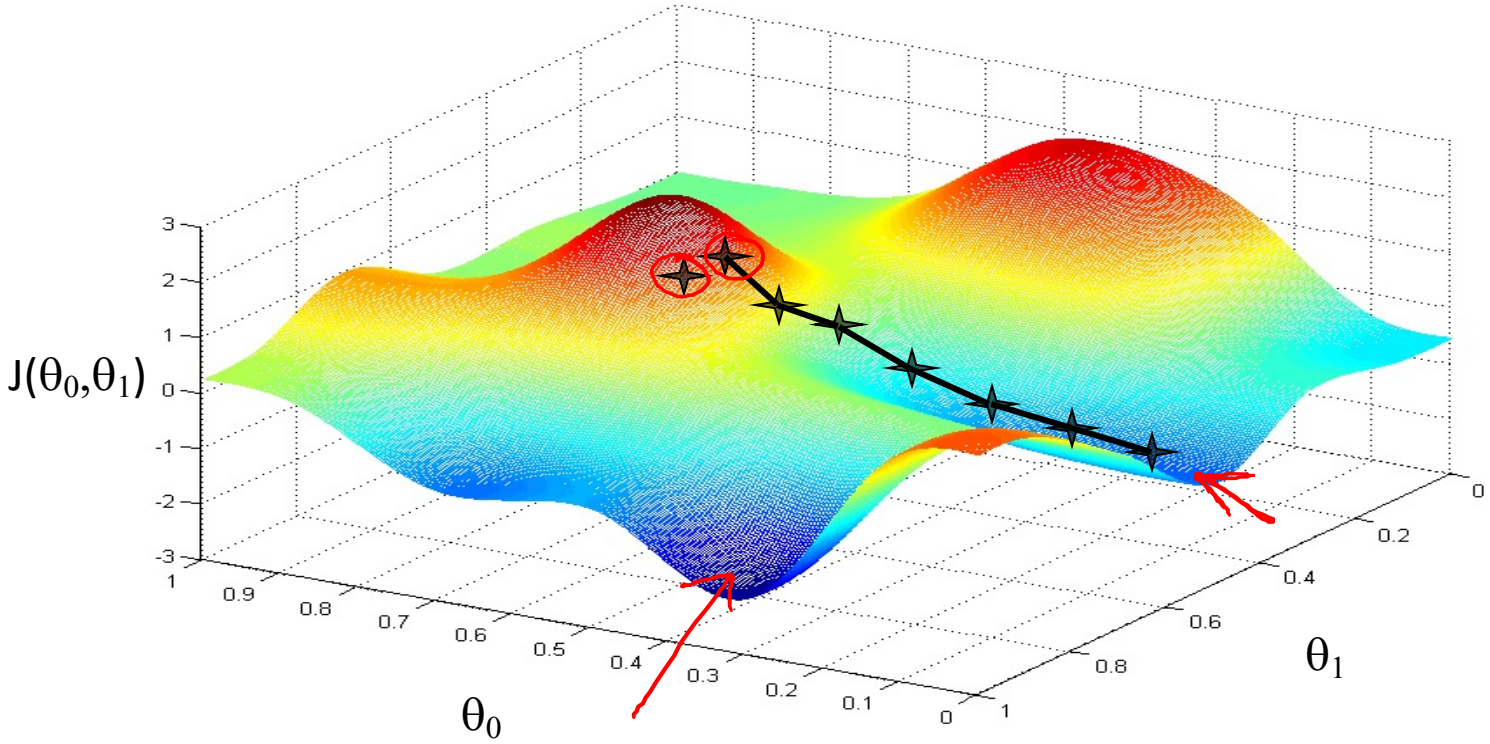
Want $\min_{\theta_0, \theta_1} J(\theta_0, \theta_1)$

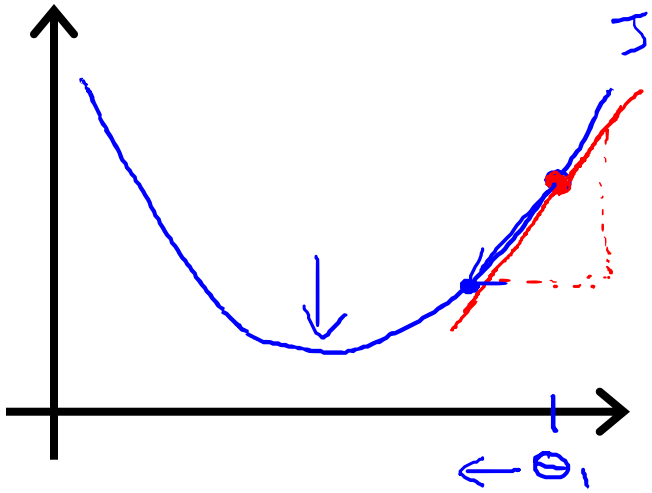
**Outline:**

- Start with some $\theta_0, \theta_1$
- Keep changing $\theta_0, \theta_1$ to reduce $J(\theta_0, \theta_1)$

  until we hopefully end up at a minimum


J(θ0,θ1)
3
2
1
0
-1
-2
-3
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
θ0
0
0.2
0.4
0.6
0.8
1
θ1


J(θ0,θ1)
3
2
1
0
-1
-2
-3
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
θ0
0
0.2
0.4
0.6
0.8
1
θ1

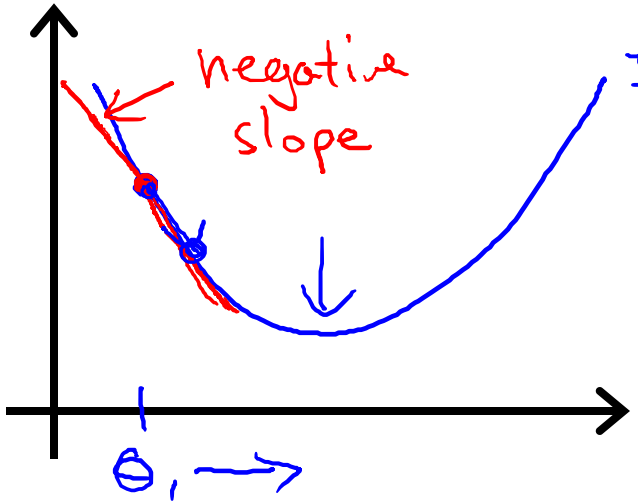



$(\theta_1 \in \mathbb{R})$

$\theta_1 := \theta_1 - \alpha \frac{d}{d\theta_1} J(\theta_1)$

$\frac{d}{d\theta_1}$ ← $\geq 0$

$\theta_1 := \theta_1 - \alpha \cdot$ (positive number)

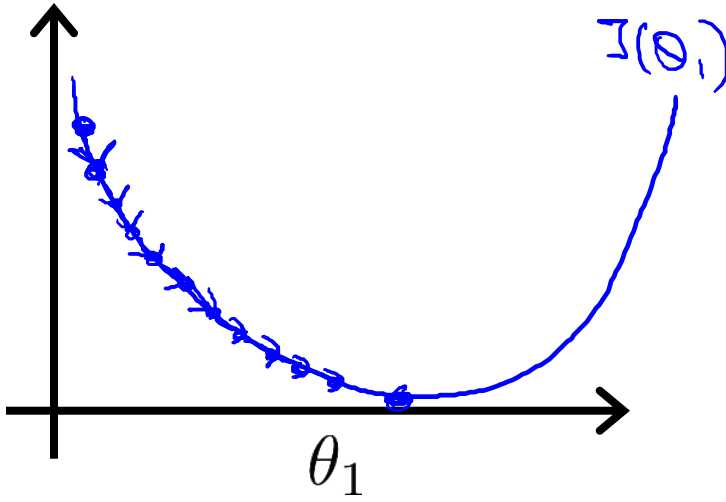



$\frac{d}{d\theta_1} J(\theta_1) \leq 0$

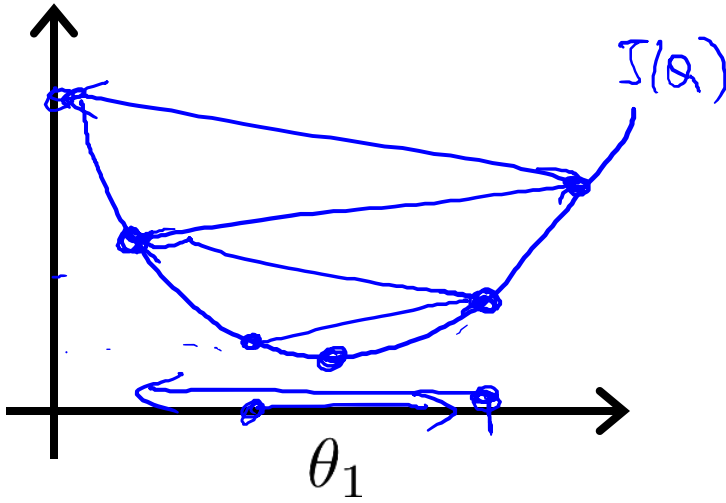
$\theta_1 := \theta_1 - \alpha$ (negative number)

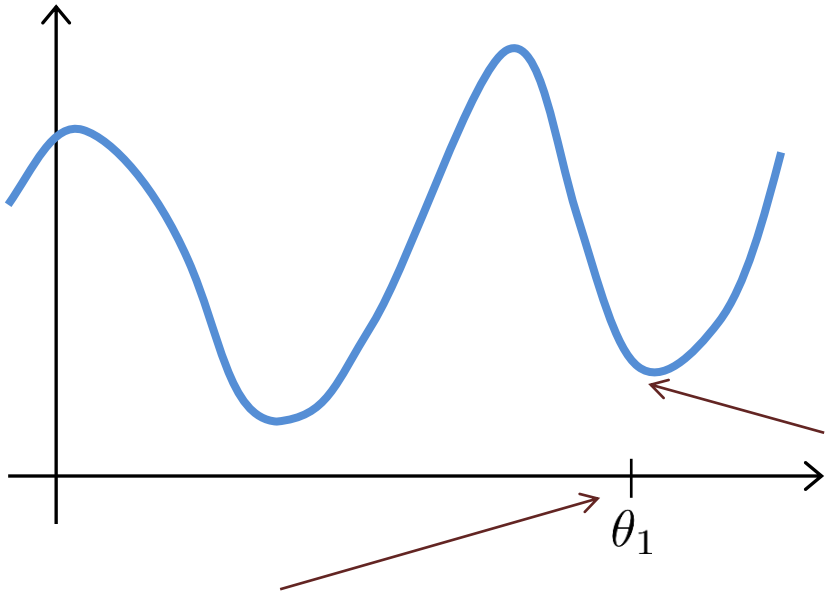
$$\theta_1 := \theta_1 - \alpha \frac{\partial}{\partial \theta_1} J(\theta_1)$$

If α is too small, gradient descent can be slow.




If α is too large, gradient descent can overshoot the minimum. It may fail to converge, or even diverge.






$$\theta_1 := \theta_1 - \alpha \frac{d}{d\theta_1} J(\theta_1)$$

# Gradient descent algorithm

repeat until convergence {

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1)$$ (simultaneously update $j = 0$ and $j = 1$)

}

# Gradient descent algorithm

repeat until convergence {

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1) \qquad (\text{for } j = 0 \text{ and } j = 1)$$

}

---

Correct: Simultaneous update

$$\text{temp0} := \theta_0 - \alpha \frac{\partial}{\partial \theta_0} J(\theta_0, \theta_1)$$
$$\text{temp1} := \theta_1 - \alpha \frac{\partial}{\partial \theta_1} J(\theta_0, \theta_1)$$
$$\theta_0 := \text{temp0}$$
$$\theta_1 := \text{temp1}$$

Incorrect:

$$\text{temp0} := \theta_0 - \alpha \frac{\partial}{\partial \theta_0} J(\theta_0, \theta_1)$$
$$\theta_0 := \text{temp0}$$
$$\text{temp1} := \theta_1 - \alpha \frac{\partial}{\partial \theta_1} J(\theta_0, \theta_1)$$
$$\theta_1 := \text{temp1}$$

## Gradient descent algorithm

repeat until convergence {

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1)$$

(for $j = 1$ and $j = 0$)

}

## Linear Regression Model

$$h_\theta(x) = \theta_0 + \theta_1 x$$

$$J(\theta_0, \theta_1) = \frac{1}{2m} \sum_{i=1}^{m} \left( h_\theta(x^{(i)}) - y^{(i)} \right)^2$$

$$\frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1) = \frac{\partial}{\partial \theta_j} \frac{1}{2m} \sum_{i=1}^{m} \left( h_\theta(x^{(i)}) - y^{(i)} \right)^2$$

$$= \frac{\partial}{\partial \theta_j} \frac{1}{2m} \sum_{i=1}^{m} \left( \theta_0 + \theta_1 x^{(i)} - y^{(i)} \right)^2$$

$$j = 0 : \frac{\partial}{\partial \theta_0} J(\theta_0, \theta_1) = \frac{1}{m} \sum_{i=1}^{m} \left( h_\theta(x^{(i)}) - y^{(i)} \right)$$

$$j = 1 : \frac{\partial}{\partial \theta_1} J(\theta_0, \theta_1) = \frac{1}{m} \sum_{i=1}^{m} \left( h_\theta(x^{(i)}) - y^{(i)} \right) \cdot x^{(i)}$$

# Gradient descent algorithm

$\frac{\partial}{\partial \theta_0} J(\theta_0, \theta_1)$

repeat until convergence {

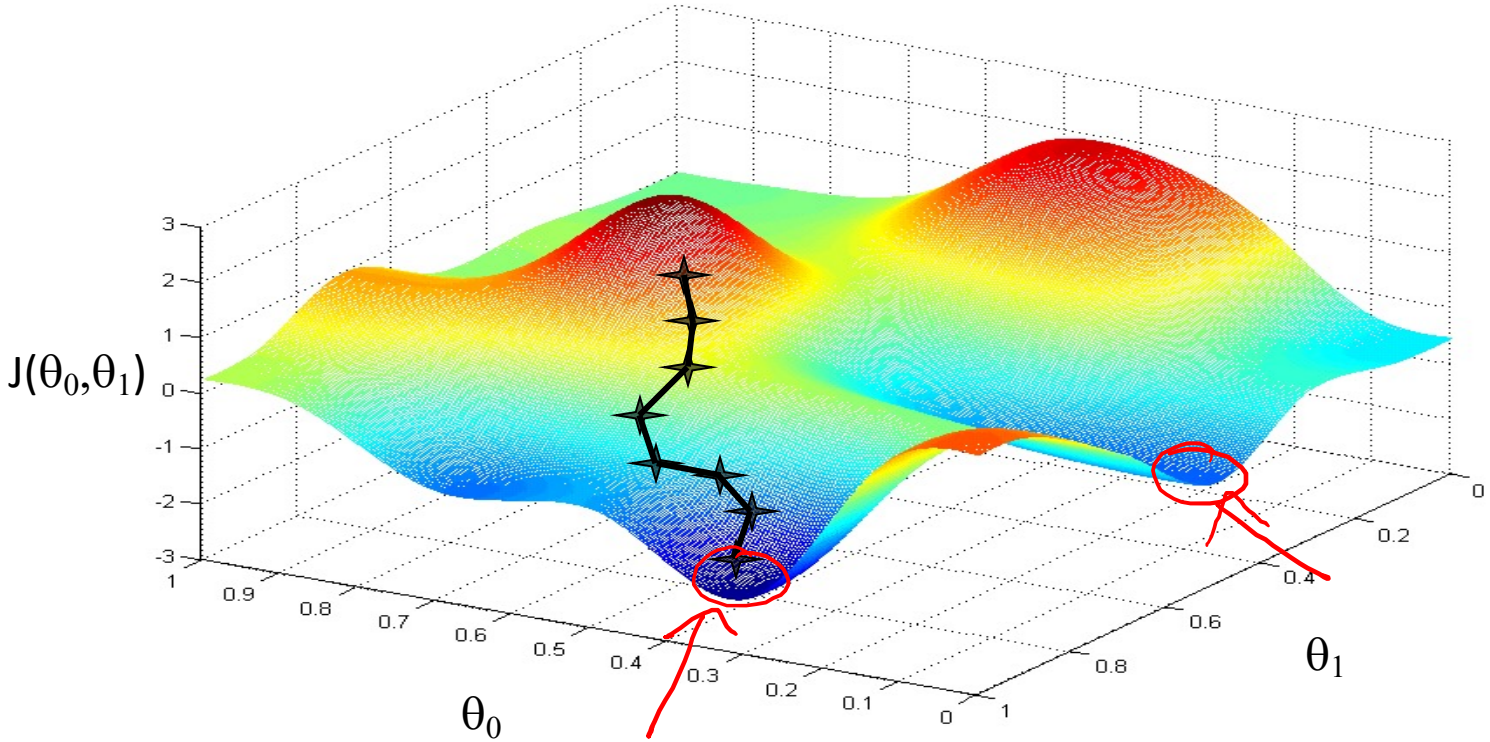
$$\theta_0 := \theta_0 - \alpha \frac{1}{m} \sum_{i=1}^{m} \left( h_\theta(x^{(i)}) - y^{(i)} \right)$$

$$\theta_1 := \theta_1 - \alpha \frac{1}{m} \sum_{i=1}^{m} \left( h_\theta(x^{(i)}) - y^{(i)} \right) \cdot x^{(i)}$$

}

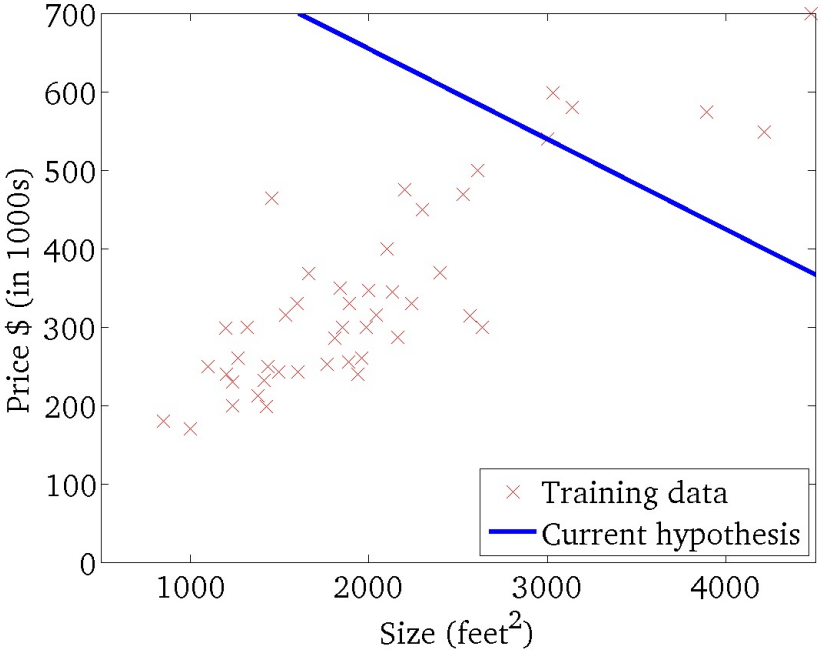
update $\theta_0$ and $\theta_1$ simultaneously

$\frac{\partial}{\partial \theta_1} J(\theta_0, \theta_1)$


J(θ0,θ1)
3
2
1
0
-1
-2
-3
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
θ0
0
0.2
0.4
0.6
0.8
1
θ1

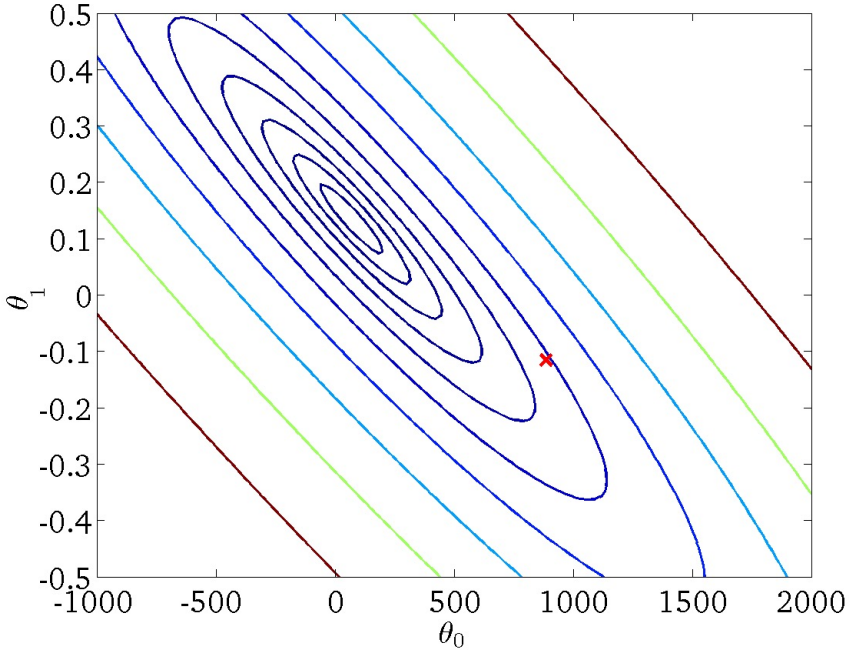
$h_\theta(x)$

(for fixed $\theta_0, \theta_1$, this is a function of x)



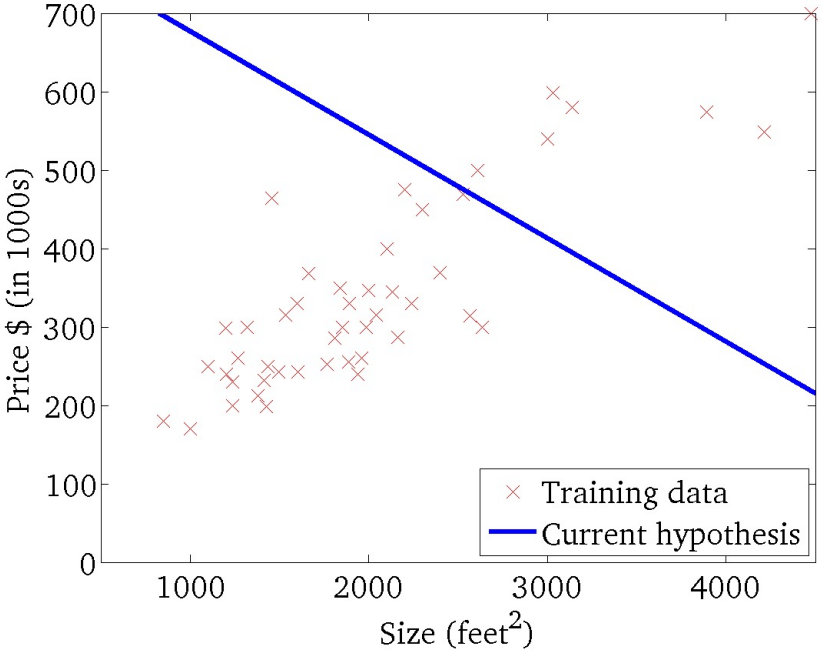

$J(\theta_0, \theta_1)$

(function of the parameters $\theta_0, \theta_1$)



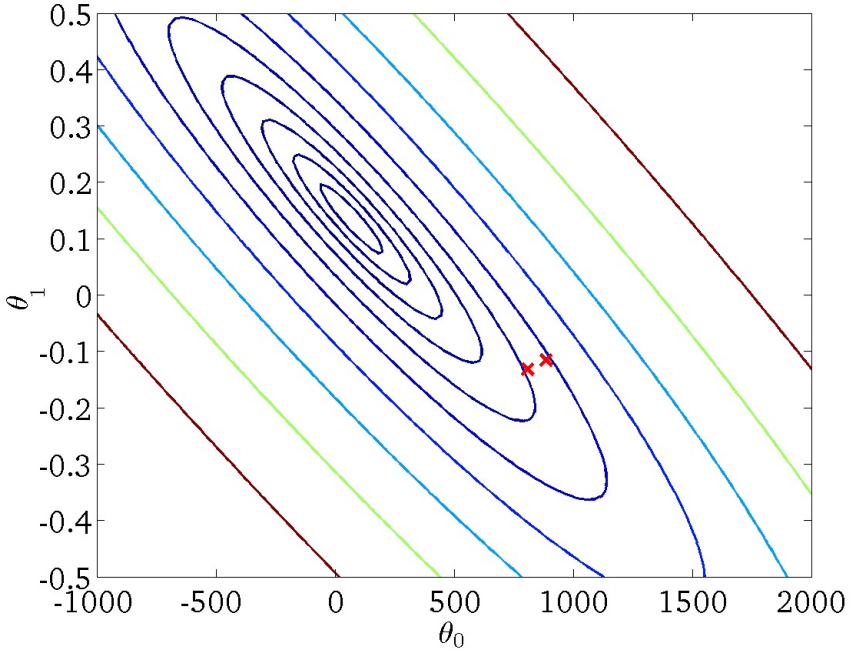
$h_\theta(x)$

(for fixed $\theta_0, \theta_1$, this is a function of x)



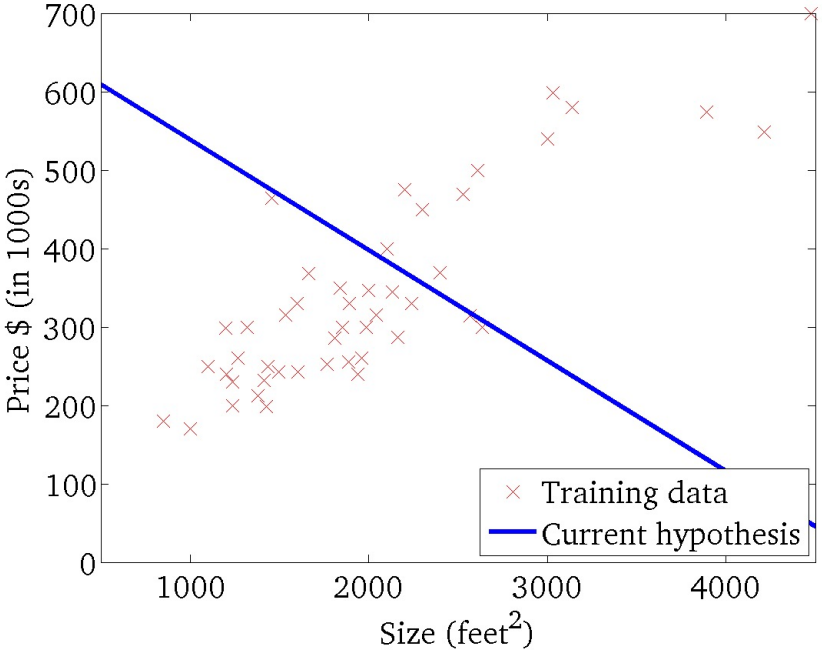

$J(\theta_0, \theta_1)$

(function of the parameters $\theta_0, \theta_1$)



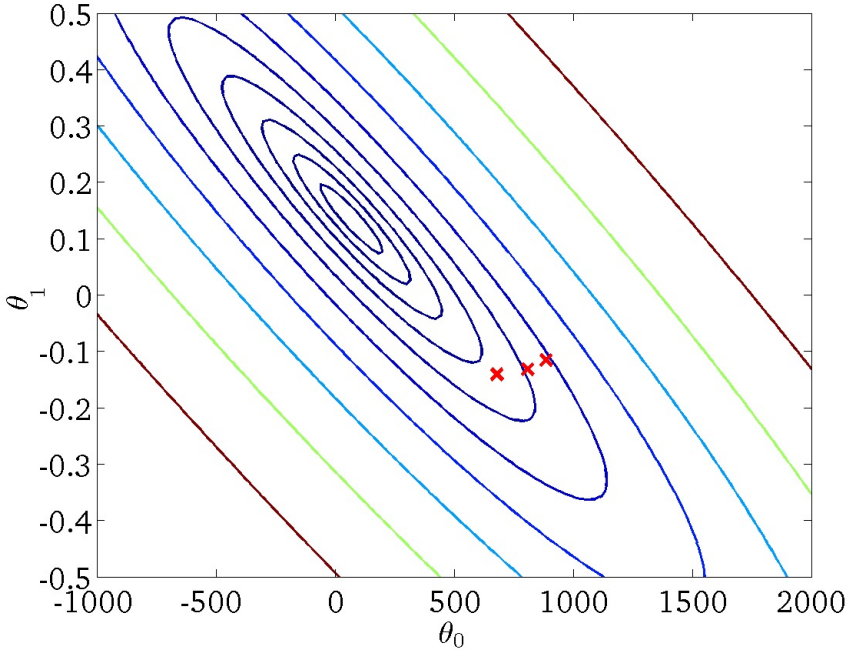
$h_\theta(x)$

(for fixed $\theta_0, \theta_1$, this is a function of x)



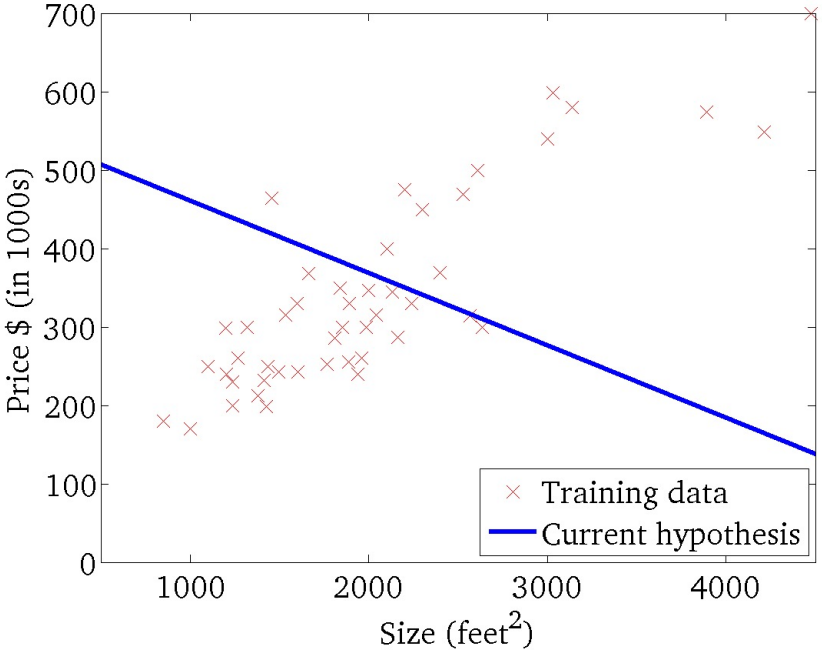

$J(\theta_0, \theta_1)$

(function of the parameters $\theta_0, \theta_1$)



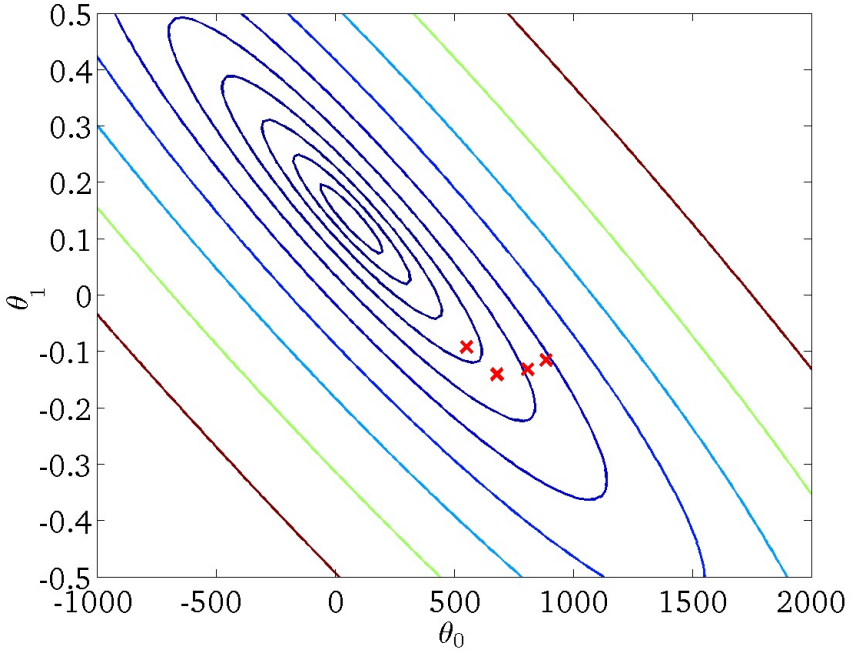
$h_\theta(x)$

(for fixed $\theta_0, \theta_1$, this is a function of x)



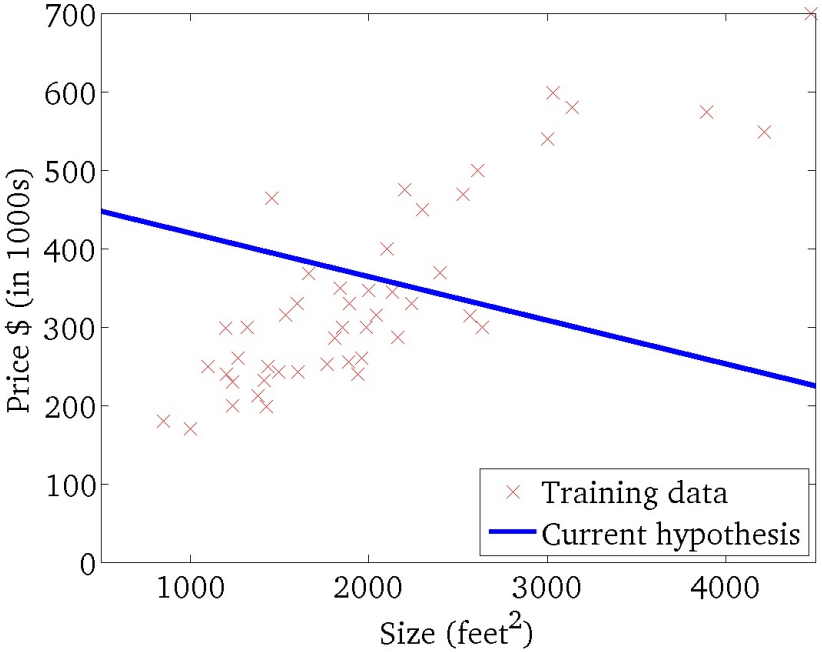

$J(\theta_0, \theta_1)$

(function of the parameters $\theta_0, \theta_1$)



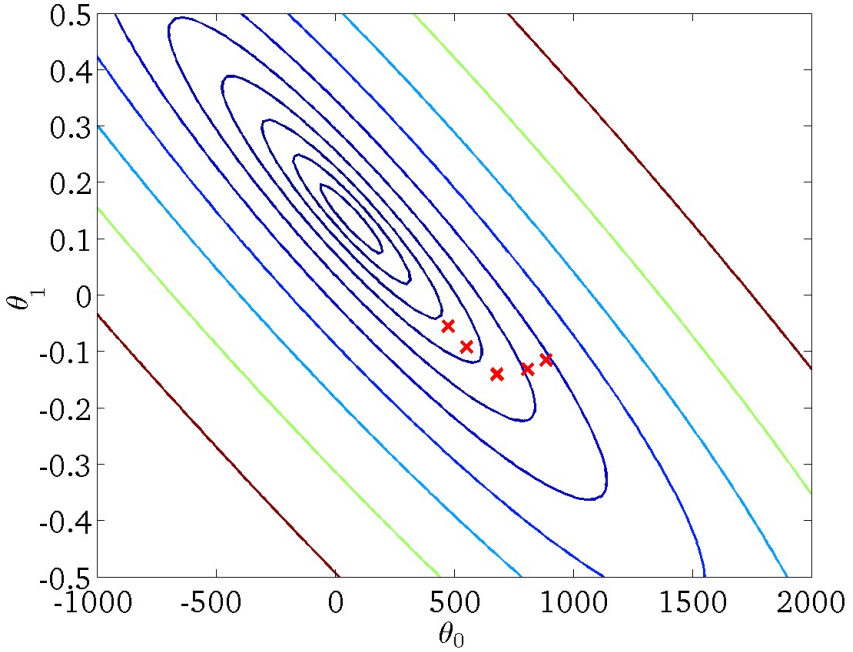
$h_\theta(x)$

(for fixed $\theta_0, \theta_1$, this is a function of x)



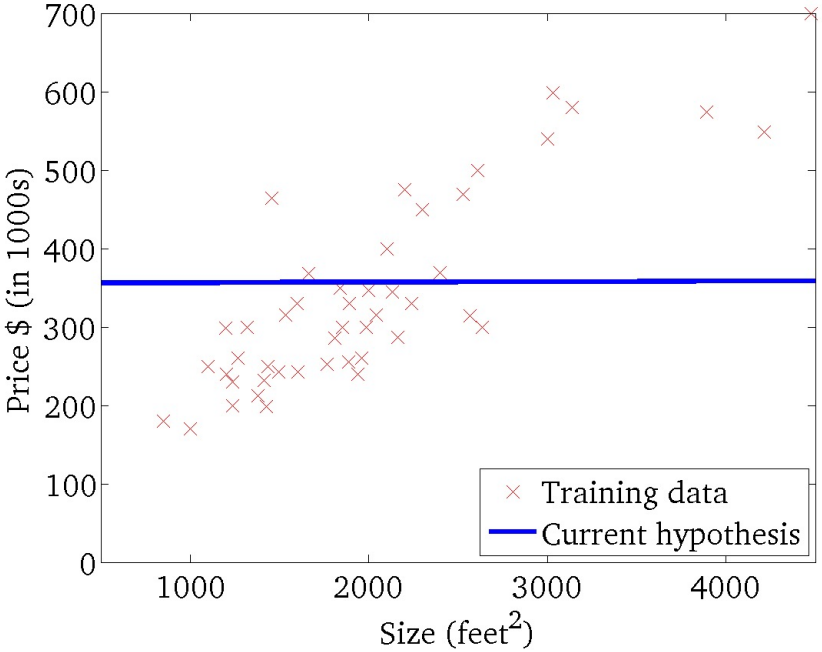

$J(\theta_0, \theta_1)$

(function of the parameters $\theta_0, \theta_1$)



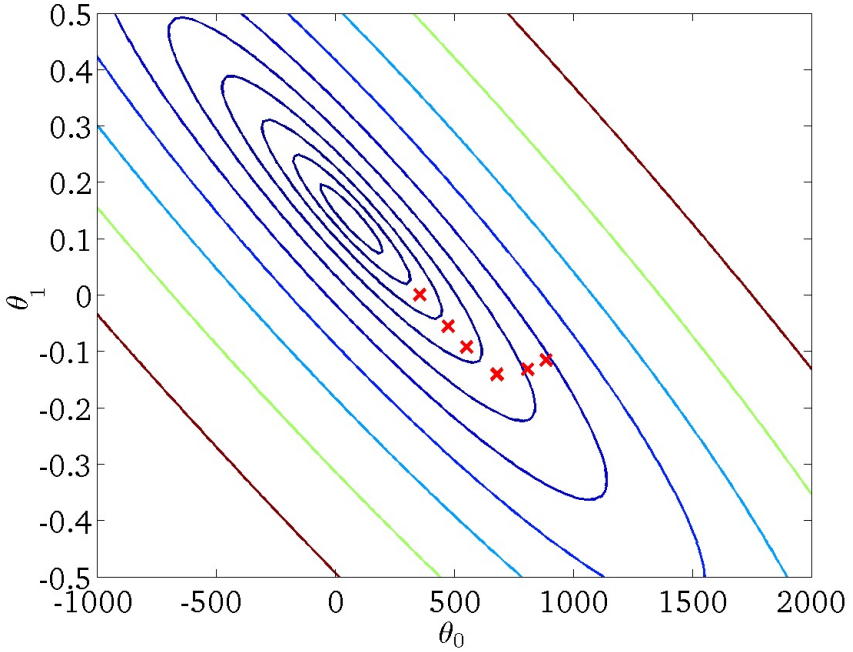
$h_\theta(x)$

(for fixed $\theta_0, \theta_1$, this is a function of x)



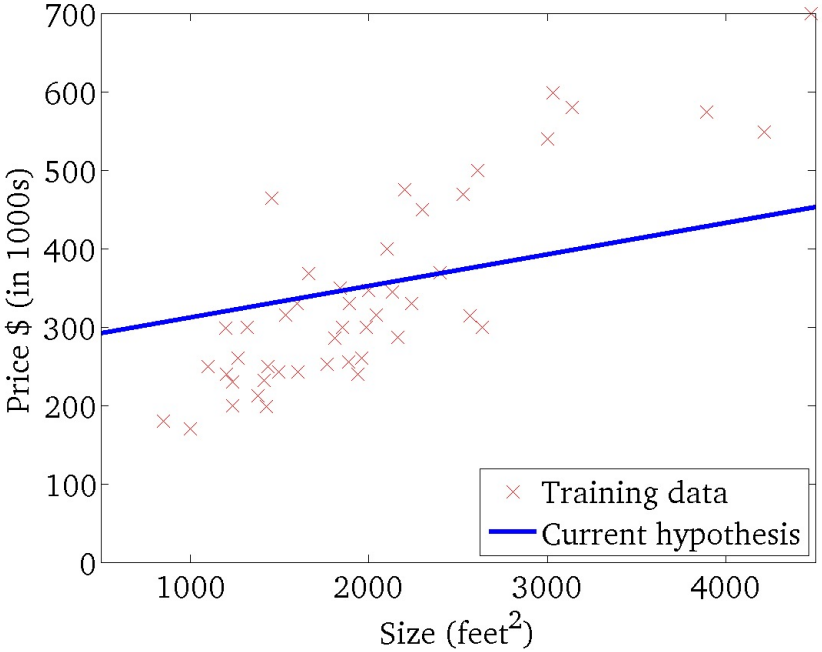

$J(\theta_0, \theta_1)$

(function of the parameters $\theta_0, \theta_1$)



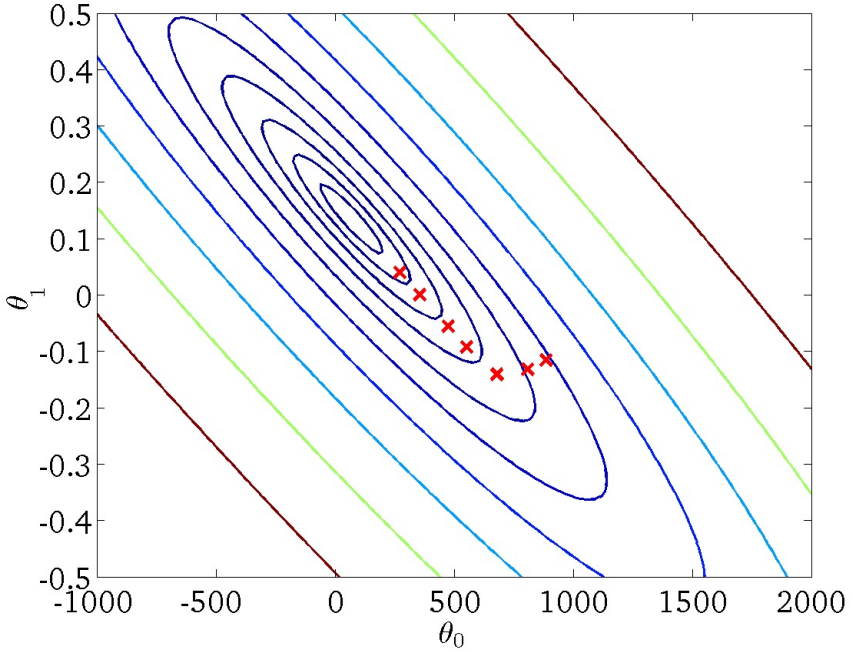
$h_\theta(x)$

(for fixed $\theta_0, \theta_1$, this is a function of x)



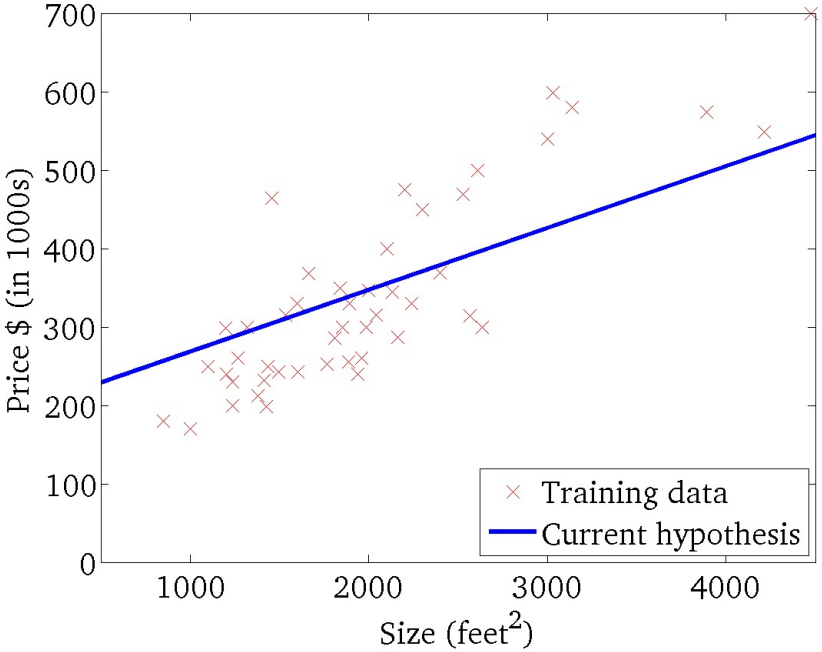

$J(\theta_0, \theta_1)$

(function of the parameters $\theta_0, \theta_1$)



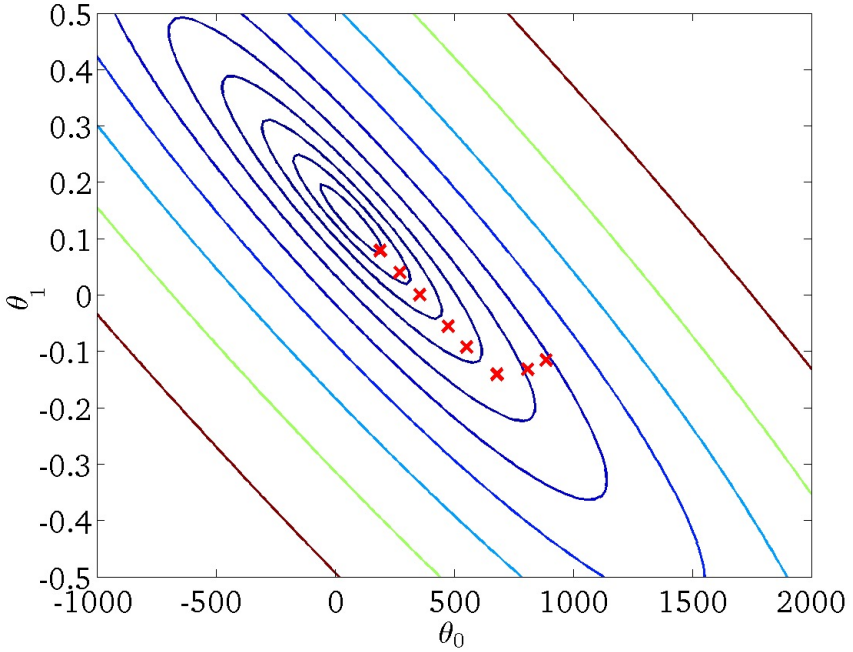
$h_\theta(x)$

(for fixed $\theta_0, \theta_1$, this is a function of x)



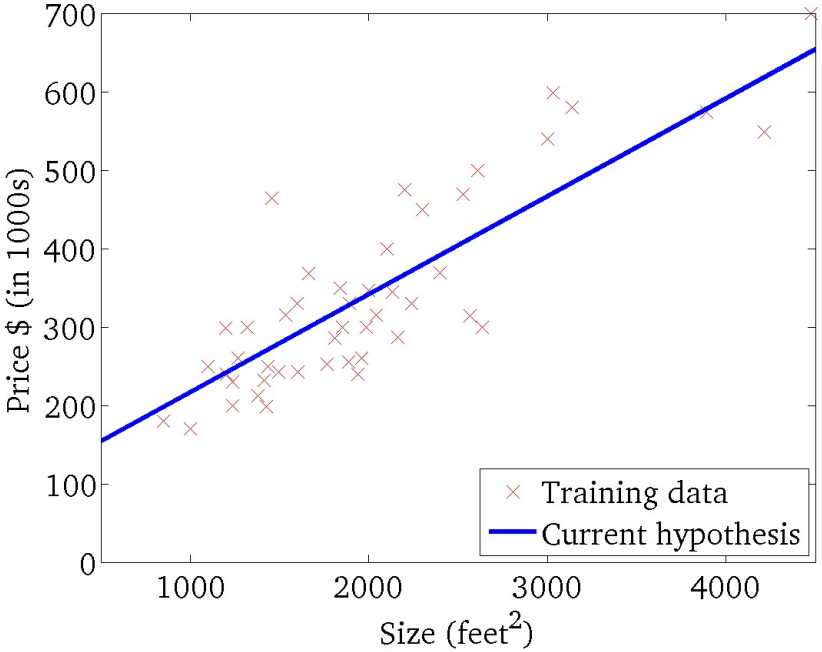

$J(\theta_0, \theta_1)$

(function of the parameters $\theta_0, \theta_1$)



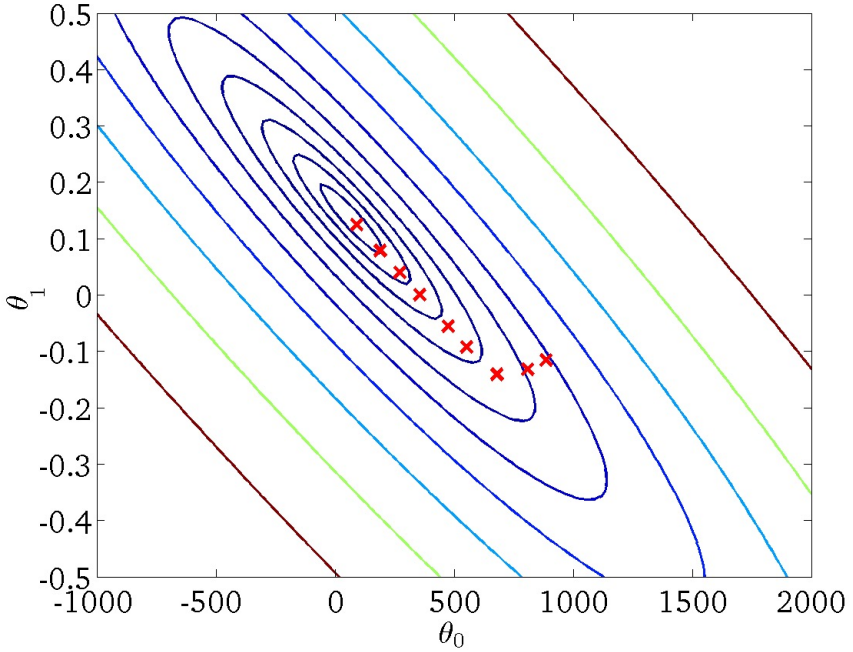
$h_\theta(x)$

(for fixed $\theta_0, \theta_1$, this is a function of x)




$J(\theta_0, \theta_1)$

(function of the parameters $\theta_0, \theta_1$)



# Linear regression with gradient descent

$$h_\theta(x) = \sum_{j=0}^{n} \theta_j x_j$$

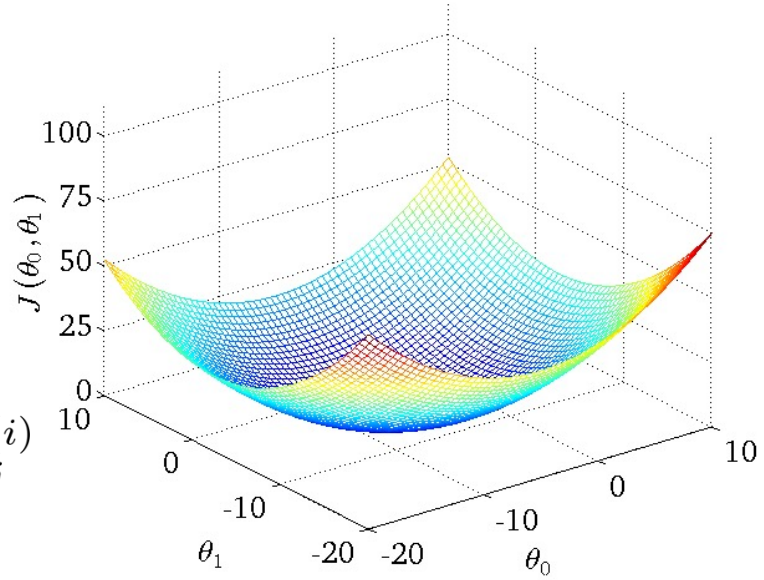
$$J_{train}(\theta) = \frac{1}{2m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})^2$$

Repeat {

$$\theta_j := \theta_j - \alpha \frac{1}{m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)}) x_j^{(i)}$$

(for every $j = 0, \ldots, n$)

}



# Linear regression with Batch Gradient Descent

$$h_\theta(x) = \sum_{j=0}^{n} \theta_j x_j$$

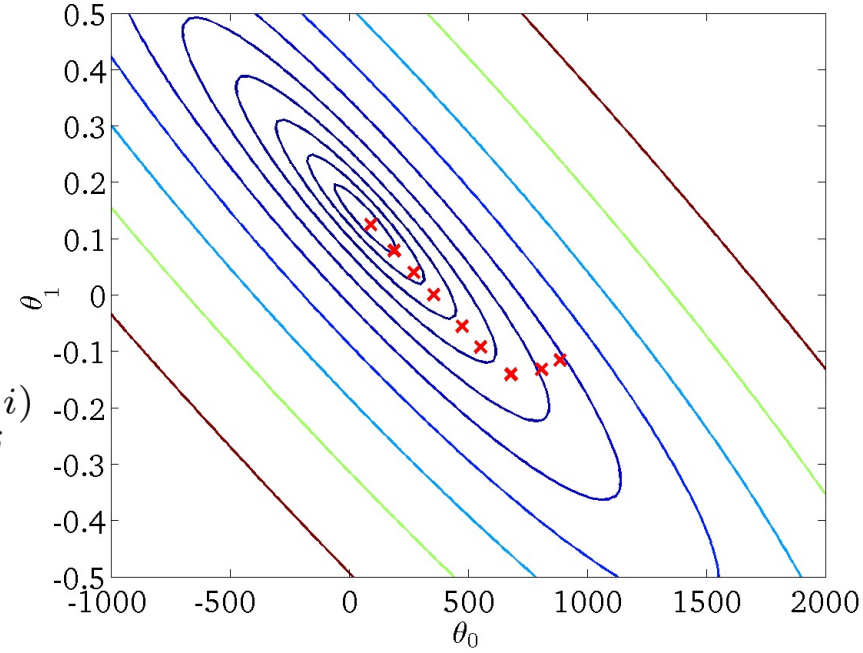
$$J_{train}(\theta) = \frac{1}{2m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})^2$$

Repeat {

$$\theta_j := \theta_j - \alpha \frac{1}{m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)}) x_j^{(i)}$$

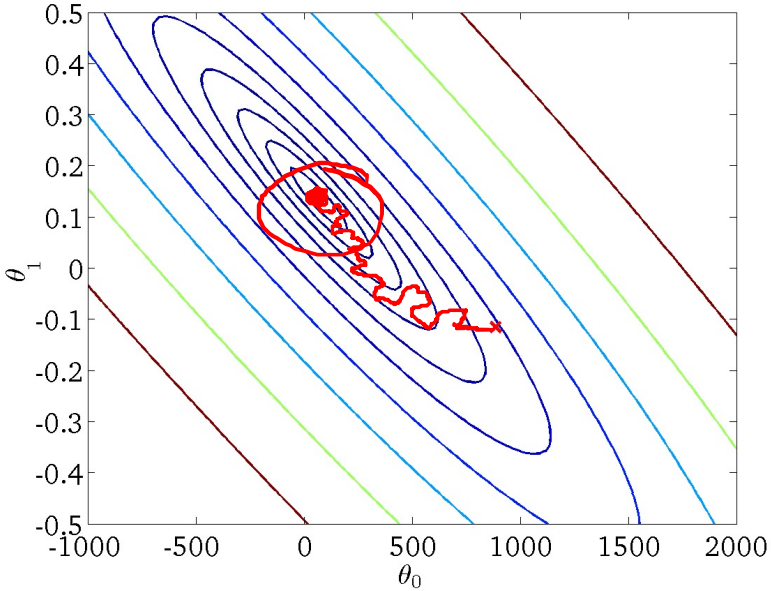
(for every $j = 0, \ldots, n$)

}



# Stochastic gradient descent

1. Randomly shuffle dataset.
2. Repeat {
   for $i := 1, \ldots, m$ {
   $\theta_j := \theta_j - \alpha(h_\theta(x^{(i)}) - y^{(i)})x_j^{(i)}$
   (for $j = 0, \ldots, n$)
   }
}




Learning rate $\alpha$ is typically held constant. Can slowly decrease $\alpha$ over time if we want $\theta$ to converge. (E.g. $\alpha = \frac{\texttt{const1}}{\texttt{iterationNumber} + \texttt{const2}}$)

# Mini-Batch Gradient Descent

Say $b = 10, m = 1000.$

Repeat {

for $i = 1, 11, 21, 31, \ldots, 991$ {

$$\theta_j := \theta_j - \alpha \frac{1}{10} \sum_{k=i}^{i+9} (h_\theta(x^{(k)}) - y^{(k)}) x_j^{(k)}$$

(for every $j = 0, \ldots, n$)

}

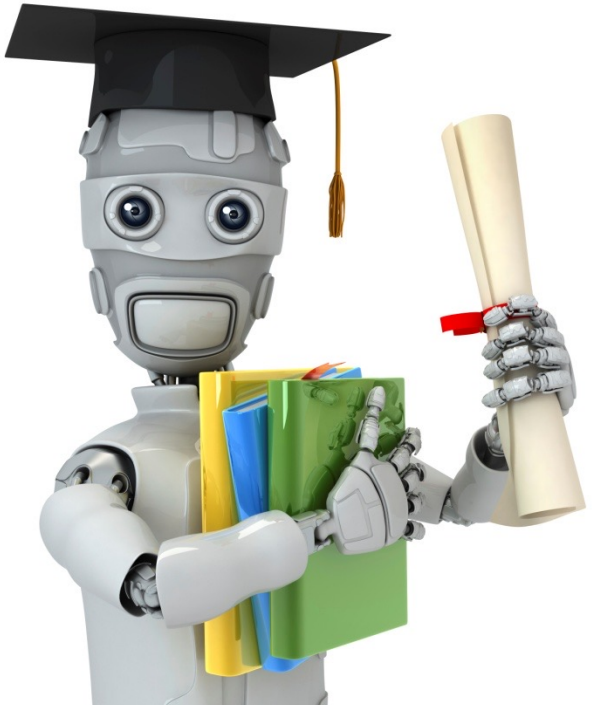
}

## Mini-batch gradient descent

Batch gradient descent: Use all $m$ examples in each iteration

Stochastic gradient descent: Use 1 example in each iteration

Mini-batch gradient descent: Use $b$ examples in each iteration

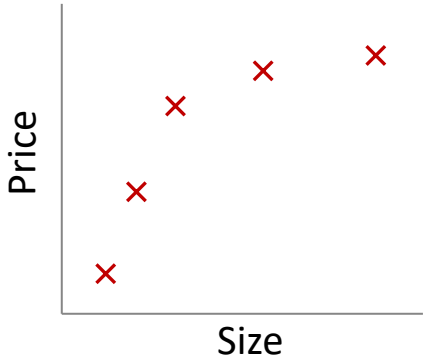


Machine Learning

# Advice for applying machine learning

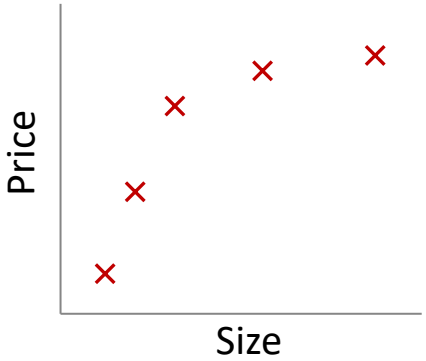
## Diagnosing bias vs. variance

# Bias/variance



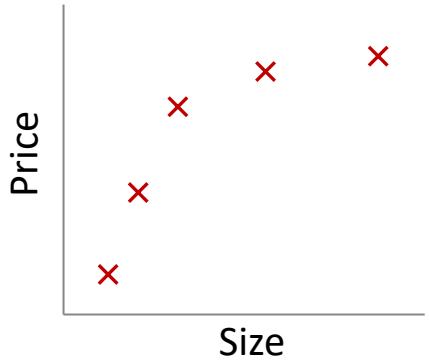

$\theta_0 + \theta_1 x$

High bias
(underfit)




$\theta_0 + \theta_1 x + \theta_2 x^2$

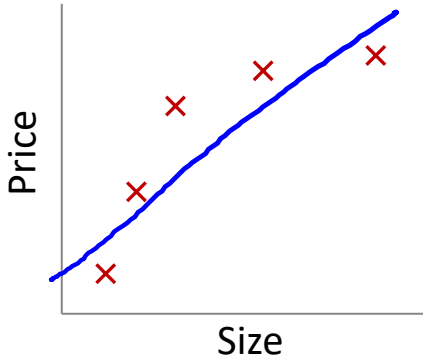
"Just right"




$\theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \theta_4 x^4$

High variance
(overfit)

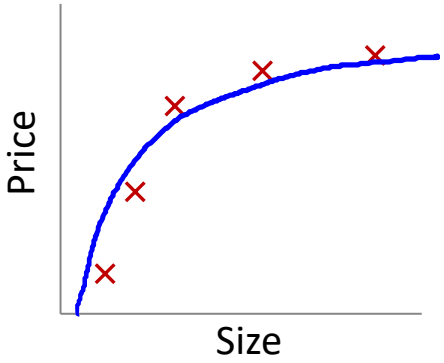
# Bias/variance




$\theta_0 + \theta_1 x$

High bias
(underfit)

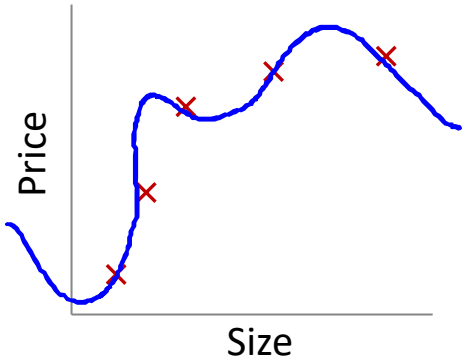
$d = 1$




$\theta_0 + \theta_1 x + \theta_2 x^2$

"Just right"

$d = 2$




$\theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \theta_4 x^4$

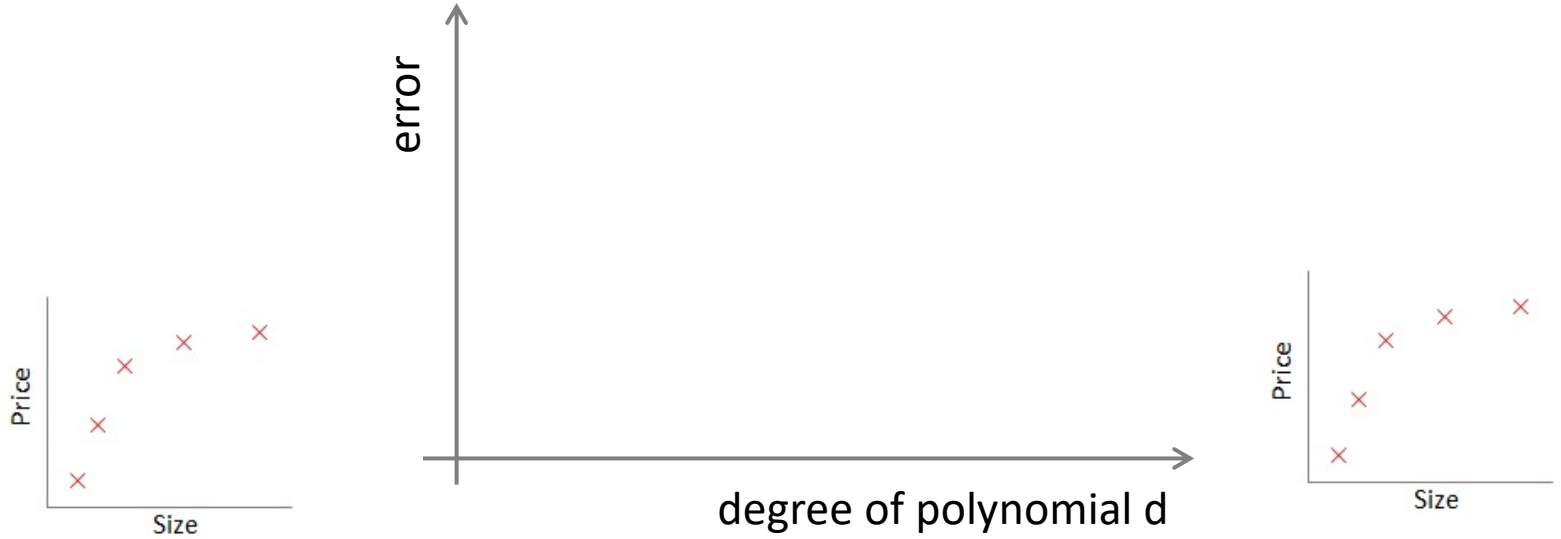
High variance
(overfit)

$d = 4$

# Bias/variance

Training error: $J_{train}(\theta) = \frac{1}{2m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})^2$

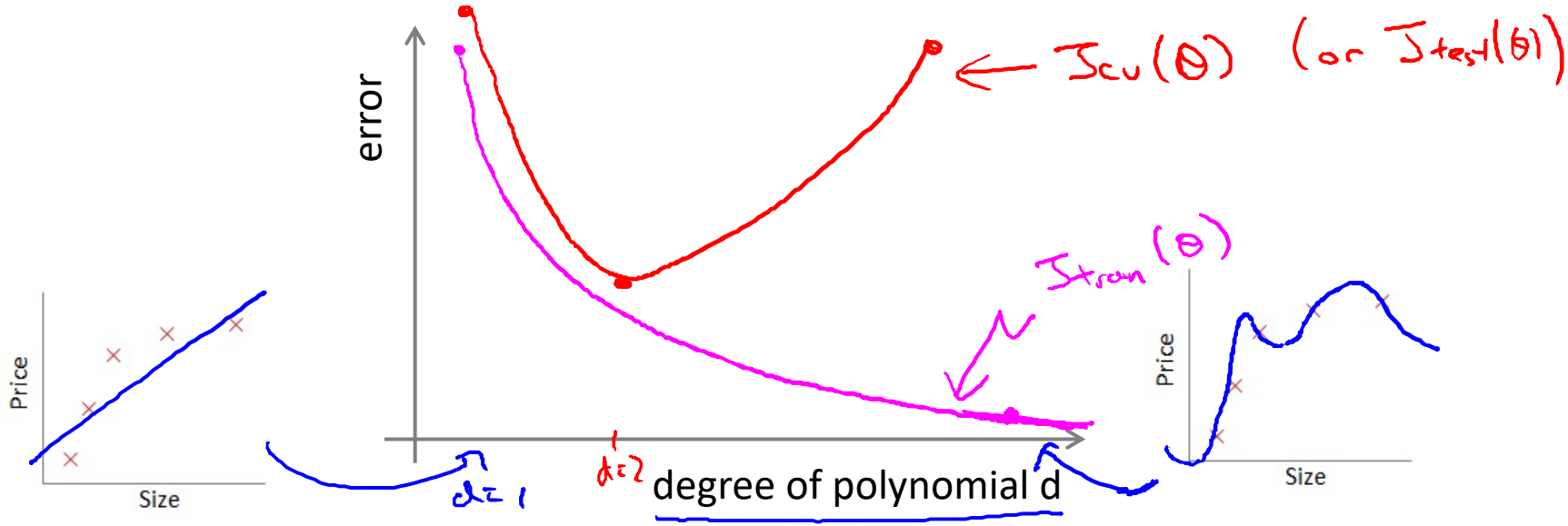
Cross validation error: $J_{cv}(\theta) = \frac{1}{2m_{cv}} \sum_{i=1}^{m_{cv}} (h_\theta(x_{cv}^{(i)}) - y_{cv}^{(i)})^2$



# Bias/variance

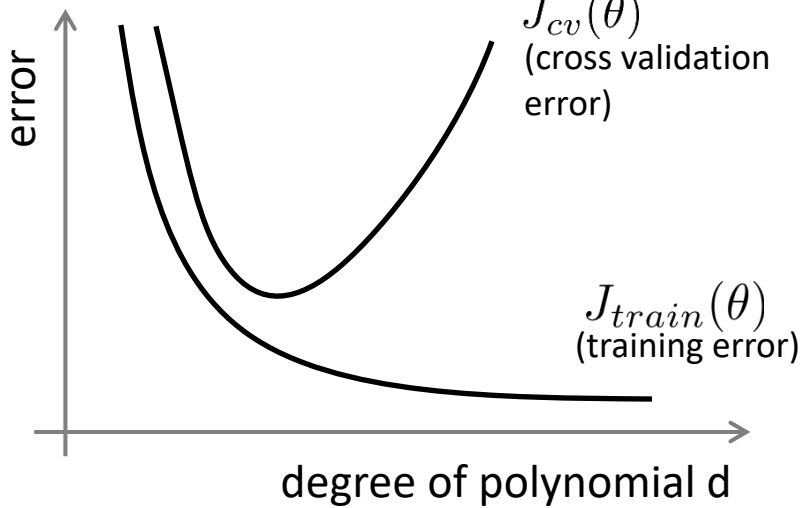
Training error: $J_{train}(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2$

Cross validation error: $J_{cv}(\theta) = \frac{1}{2m_{cv}}\sum_{i=1}^{m_{cv}}(h_\theta(x_{cv}^{(i)}) - y_{cv}^{(i)})^2$ (or $J_{test}(\theta)$)



## Diagnosing bias vs. variance

Suppose your learning algorithm is performing less well than you were hoping. ($J_{cv}(\theta)$ or $J_{test}(\theta)$ is high.) Is it a bias problem or a variance problem?

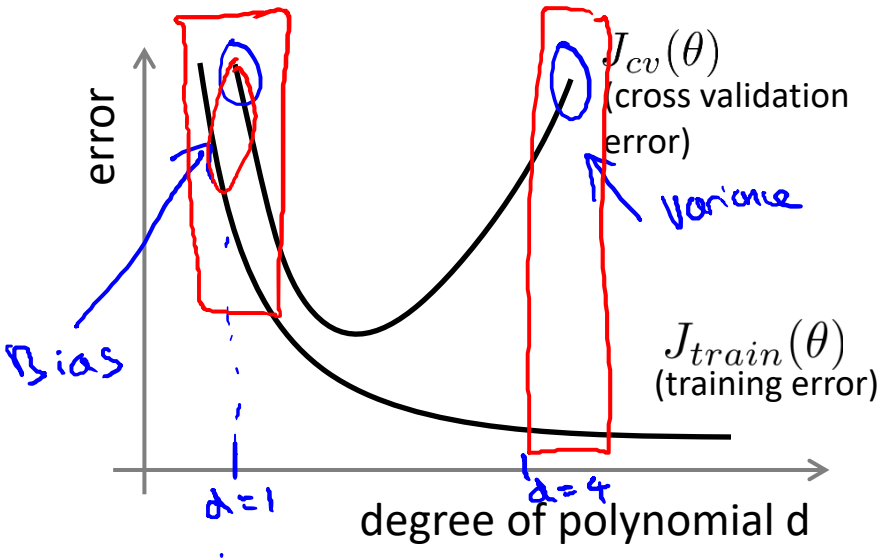



Bias (underfit):

Variance (overfit):

## Diagnosing bias vs. variance

Suppose your learning algorithm is performing less well than you were hoping. ($J_{cv}(\theta)$ or $J_{test}(\theta)$ is high.) Is it a bias problem or a variance problem?




Bias (underfit):

→ $J_{train}(\theta)$ will be high

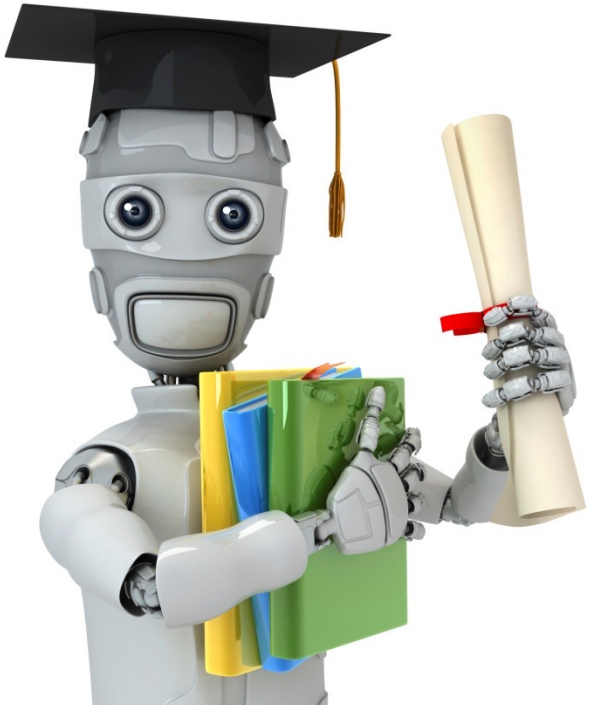
$J_{cv}(\theta) \approx J_{train}(\theta)$

Variance (overfit):

→ $J_{train}(\theta)$ will be low

$J_{cv}(\theta) \gg J_{train}(\theta)$

>>



Machine Learning

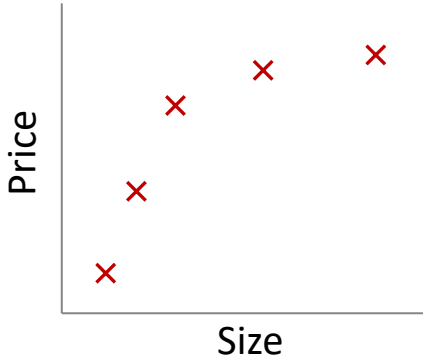
# Advice for applying machine learning

## Regularization and bias/variance

# Linear regression with regularization

Model: $h_\theta(x) = \theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \theta_4 x^4$

$$J(\theta) = \frac{1}{2m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})^2 + \frac{\lambda}{2m} \sum_{j=1}^{m} \theta_j^2$$

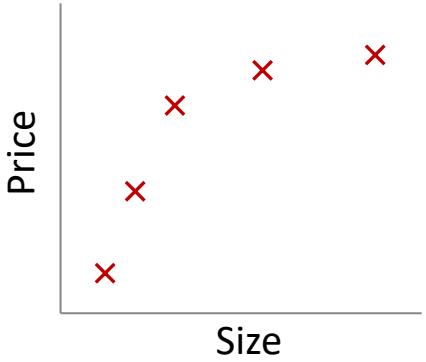



Large $\lambda$

High bias (underfit)

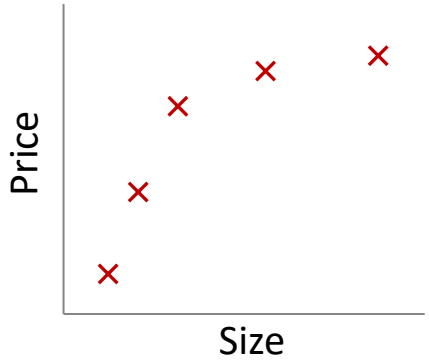
$\lambda = 10000.\ \theta_1 \approx 0, \theta_2 \approx 0, \ldots$

$h_\theta(x) \approx \theta_0$




Intermediate $\lambda$

"Just right"



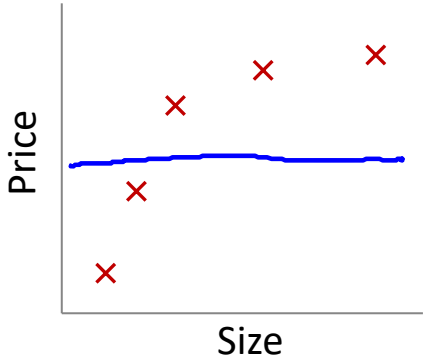

Small $\lambda$

High variance (overfit)

# Linear regression with regularization

Model: $h_\theta(x) = \theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \theta_4 x^4$

$$J(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2 + \frac{\lambda}{2m}\sum_{j=1}^{m}\theta_j^2$$

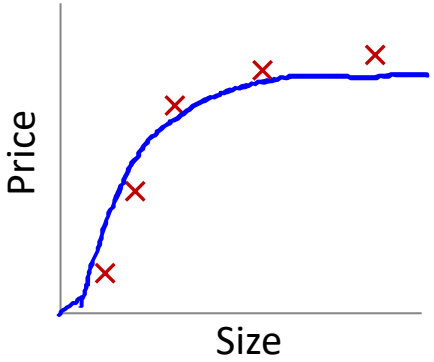



Large $\lambda$

High bias (underfit)

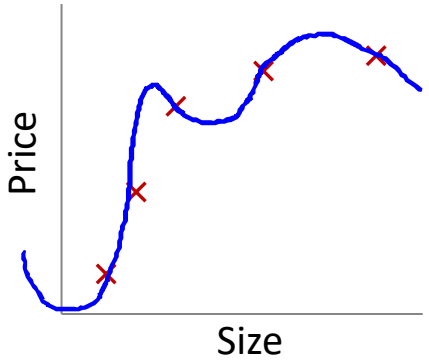
$\lambda = 10000.\ \theta_1 \approx 0, \theta_2 \approx 0, \ldots$

$h_\theta(x) \approx \theta_0$




Intermediate $\lambda$

"Just right"




Small $\lambda$

High variance (overfit)

$\lambda = 0$

# Choosing the regularization parameter $\lambda$

$$h_\theta(x) = \theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \theta_4 x^4$$

$$J(\theta) = \frac{1}{2m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})^2 + \frac{\lambda}{2m} \sum_{j=1}^{m} \theta_j^2$$

$$J_{train}(\theta) = \frac{1}{2m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})^2$$

$$J_{cv}(\theta) = \frac{1}{2m_{cv}} \sum_{i=1}^{m_{cv}} (h_\theta(x_{cv}^{(i)}) - y_{cv}^{(i)})^2$$

$$J_{test}(\theta) = \frac{1}{2m_{test}} \sum_{i=1}^{m_{test}} (h_\theta(x_{test}^{(i)}) - y_{test}^{(i)})^2$$

# Choosing the regularization parameter $\lambda$

$$h_\theta(x) = \theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \theta_4 x^4$$

$$J(\theta) = \frac{1}{2m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})^2 + \frac{\lambda}{2m} \sum_{j=1}^{m} \theta_j^2$$

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$$J_{test}(\theta) = \frac{1}{2m_{test}} \sum_{i=1}^{m_{test}} (h_\theta(x_{test}^{(i)}) - y_{test}^{(i)})^2$$

$J(\theta)$

$J_{train}$

$J_{cv}$

$J_{test}$

## Choosing the regularization parameter $\lambda$

Model: $h_\theta(x) = \theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \theta_4 x^4$

$$J(\theta) = \frac{1}{2m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})^2 + \frac{\lambda}{2m} \sum_{j=1}^{m} \theta_j^2$$

$\lambda = 0$

$\lambda = 0.01$

$\lambda = 0.02$

$\lambda = 0.04$

$\lambda = 0.08$

$\vdots$

$\lambda = 10$

Pick (say) $\theta^{(5)}$. Test error:

# Choosing the regularization parameter $\lambda$

Model: $h_\theta(x) = \theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \theta_4 x^4$

$$J(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2 + \frac{\lambda}{2m}\sum_{j=1}^{m}\theta_j^2$$

1. Try $\lambda = 0$ → $\min_\theta J(\theta)$ → $\theta^{(1)}$ → $J_{cv}(\theta^{(1)})$
2. Try $\lambda = 0.01$ → $\min_\theta J(\theta)$ → $\theta^{(2)}$ → $J_{cv}(\theta^{(2)})$
3. Try $\lambda = 0.02$ → $\theta^{(3)}$ → $J_{cv}(\theta^{(3)})$
4. Try $\lambda = 0.04$
5. Try $\lambda = 0.08$ → $\theta^{(5)}$ $J_{cv}(\theta^{(5)})$

⋮

12. Try $\lambda = 10$ → $\theta^{(12)}$ → $J_{cv}(\theta^{(12)})$

$10 \cdot 24$

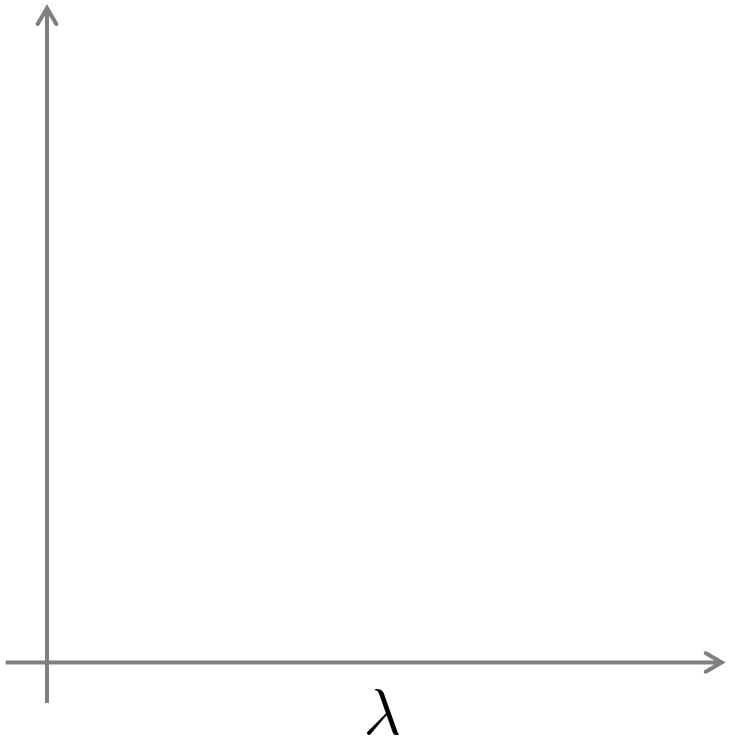
Pick (say) $\theta^{(5)}$. Test error: $J_{test}(\theta^{(5)})$

# Bias/variance as a function of the regularization parameter $\lambda$

$$J(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2 + \frac{\lambda}{2m}\sum_{j=1}^{m}\theta_j^2$$

$$J_{train}(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2$$

$$J_{cv}(\theta) = \frac{1}{2m_{cv}}\sum_{i=1}^{m_{cv}}(h_\theta(x_{cv}^{(i)}) - y_{cv}^{(i)})^2$$



# Bias/variance as a function of the regularization parameter $\lambda$

$$J(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2 + \frac{\lambda}{2m}\sum_{j=1}^{m}\theta_j^2$$

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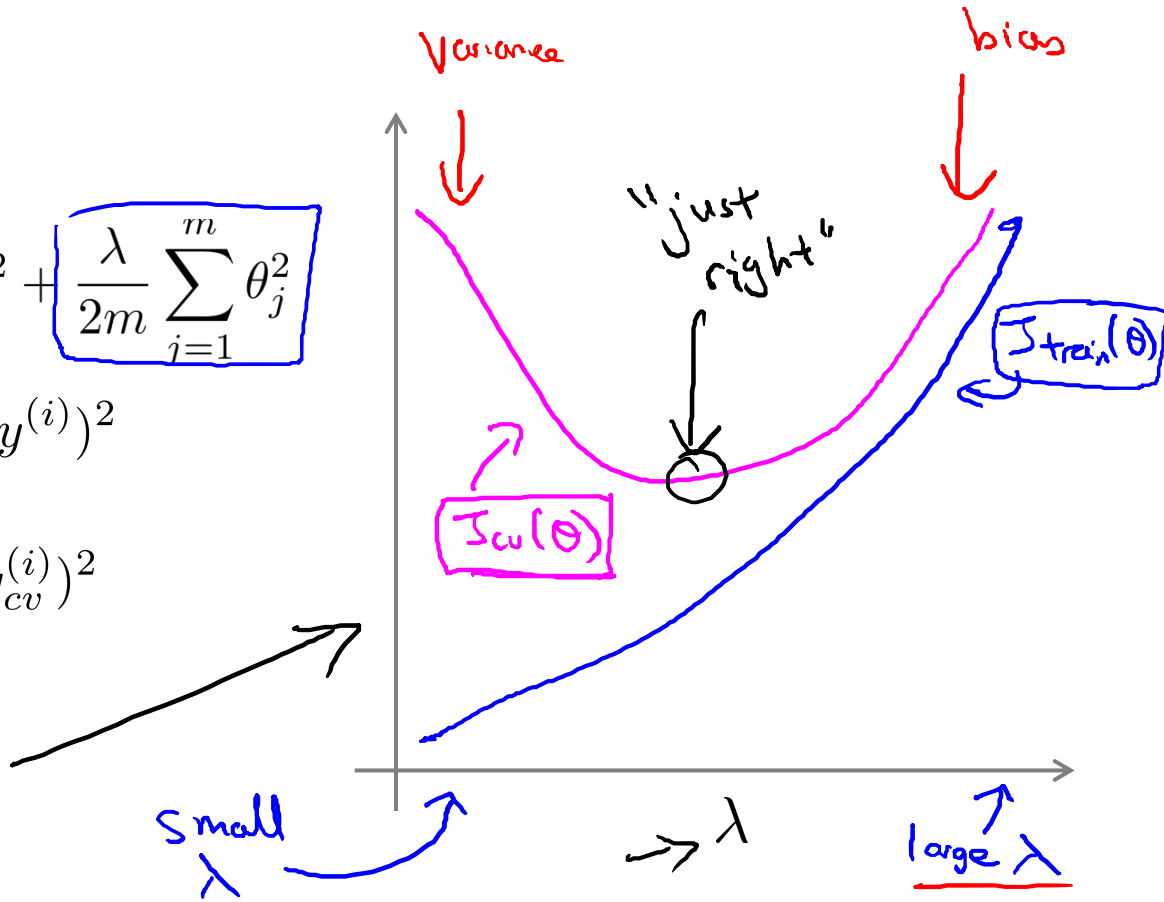
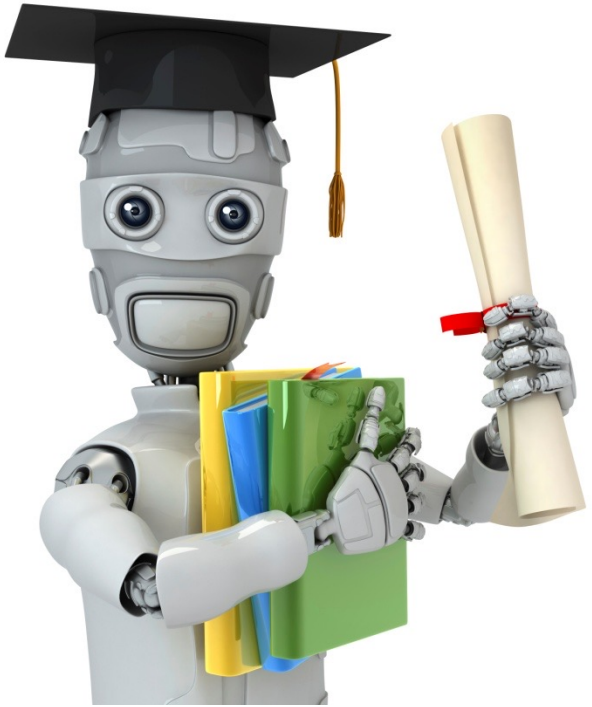




Machine Learning

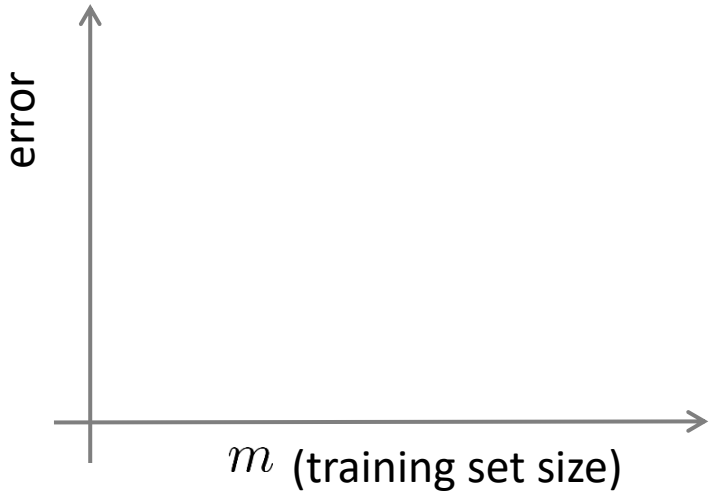
# Advice for applying machine learning

## Learning curves

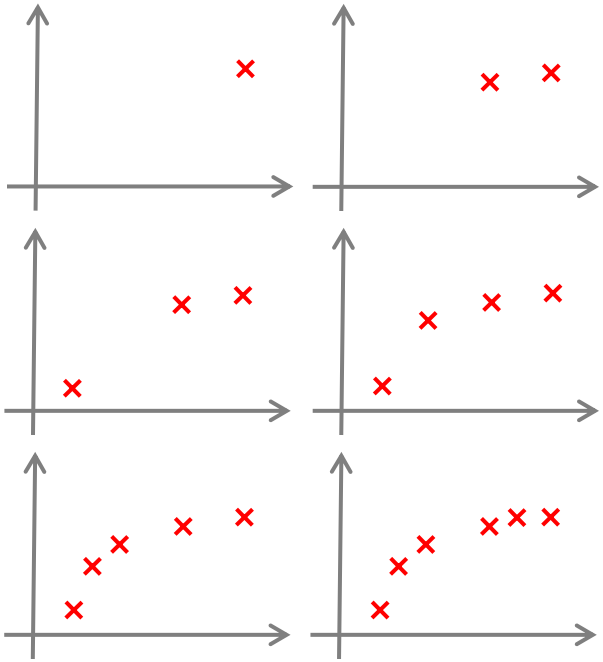
# Learning curves

$$J_{train}(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2$$

$$J_{cv}(\theta) = \frac{1}{2m_{cv}}\sum_{i=1}^{m_{cv}}(h_\theta(x_{cv}^{(i)}) - y_{cv}^{(i)})^2$$



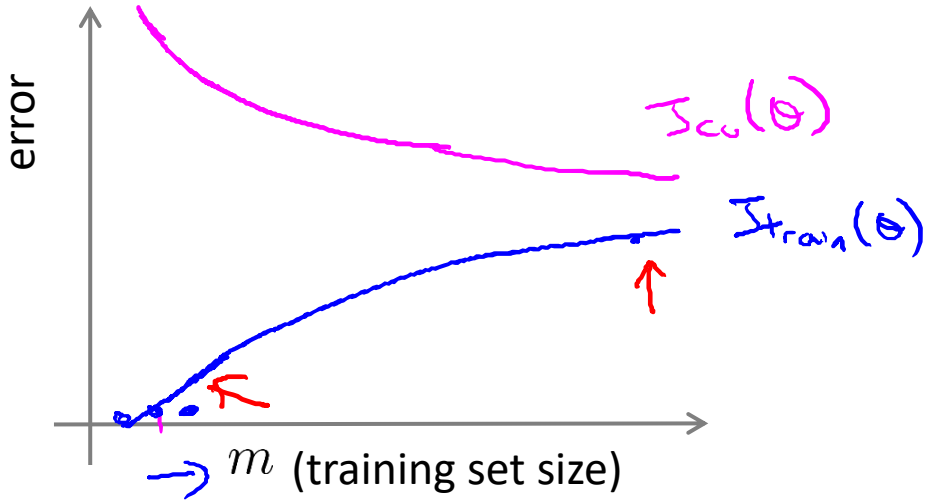

$$h_\theta(x) = \theta_0 + \theta_1 x + \theta_2 x^2$$



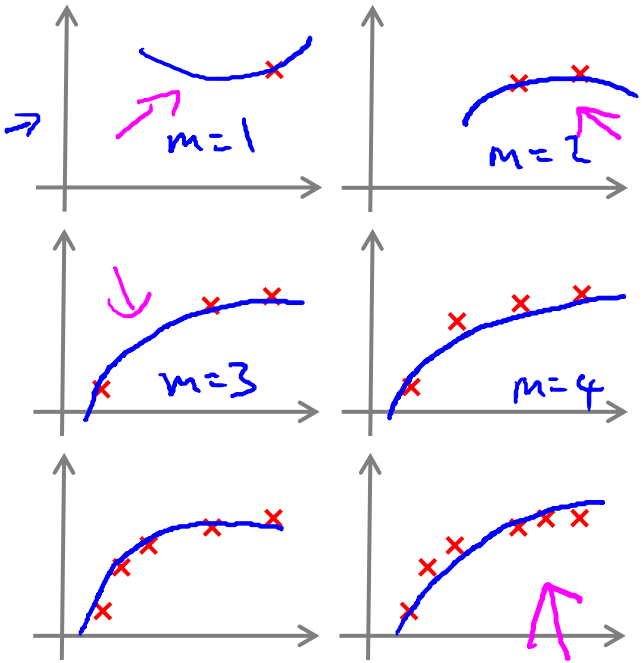
# Learning curves

$$J_{train}(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2$$

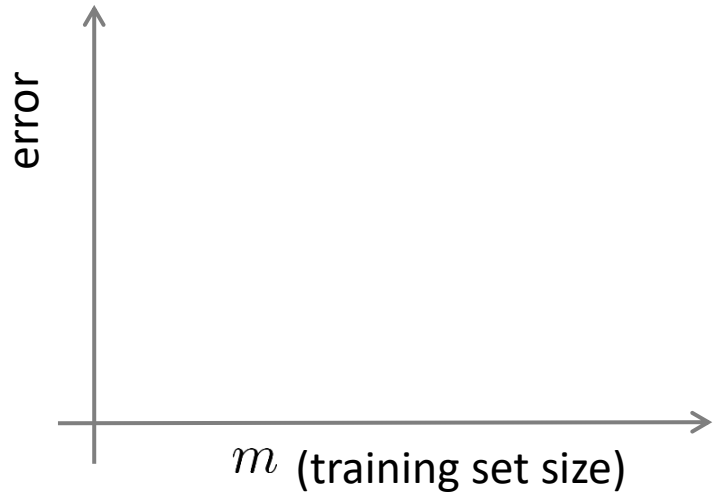
$$J_{cv}(\theta) = \frac{1}{2m_{cv}}\sum_{i=1}^{m_{cv}}(h_\theta(x_{cv}^{(i)}) - y_{cv}^{(i)})^2$$




$$h_\theta(x) = \theta_0 + \theta_1 x + \theta_2 x^2$$

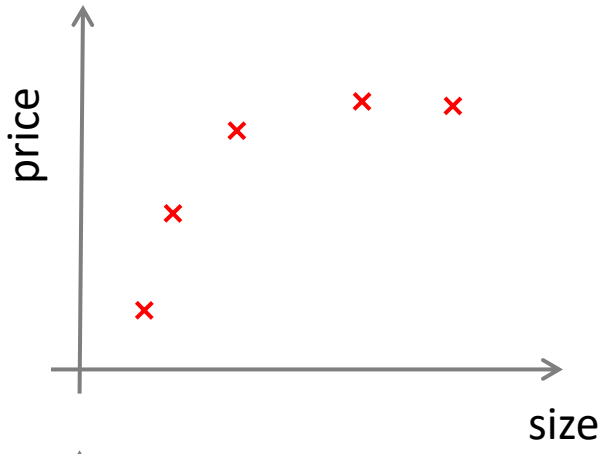


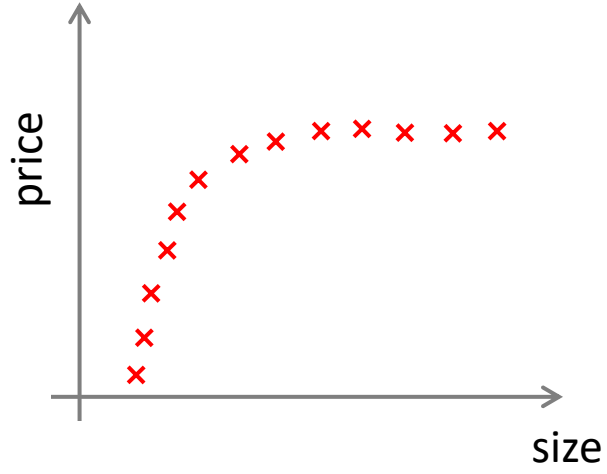
## High bias



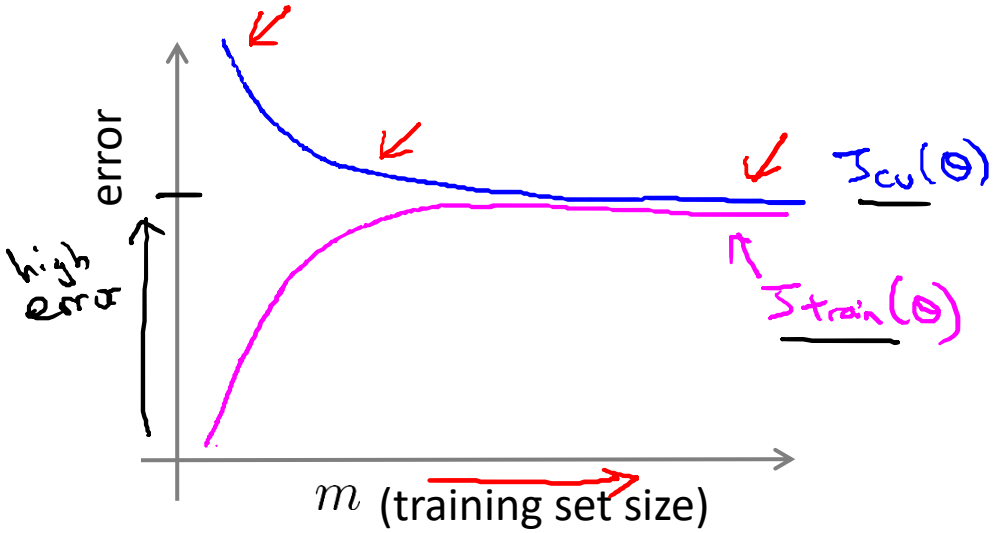

If a learning algorithm is suffering from high bias, getting more training data will not (by itself) help much.

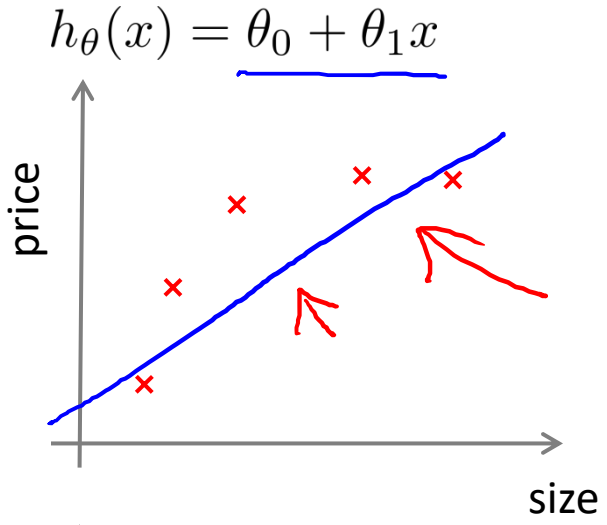
$h_\theta(x) = \theta_0 + \theta_1 x$

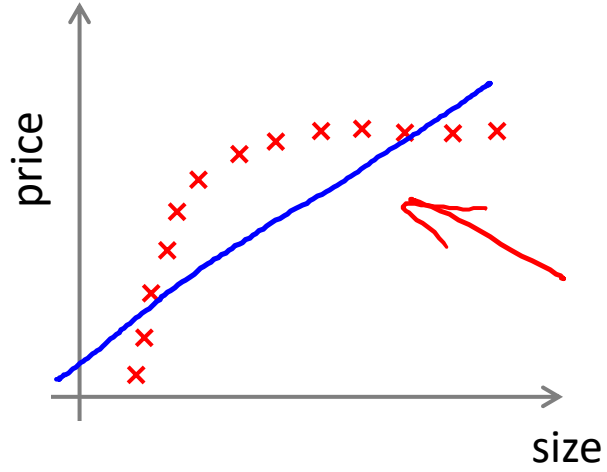


## High bias

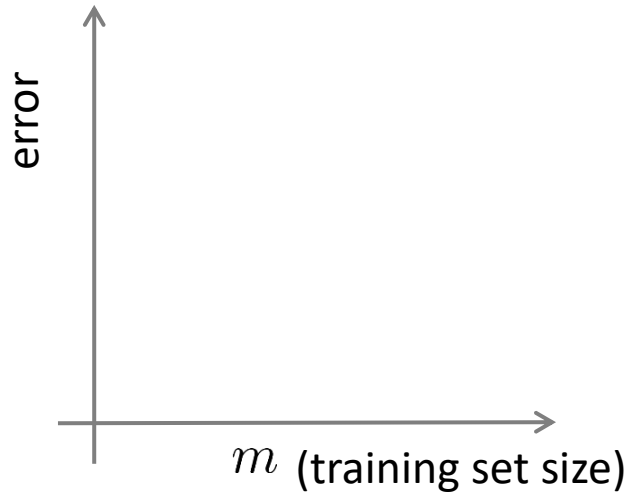



If a learning algorithm is suffering from high bias, getting more training data will not (by itself) help much.

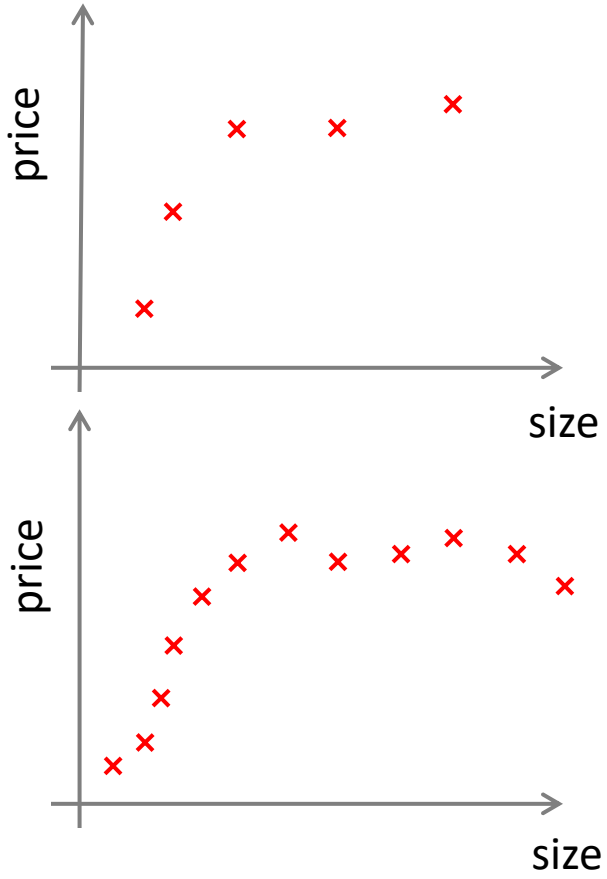
## High variance



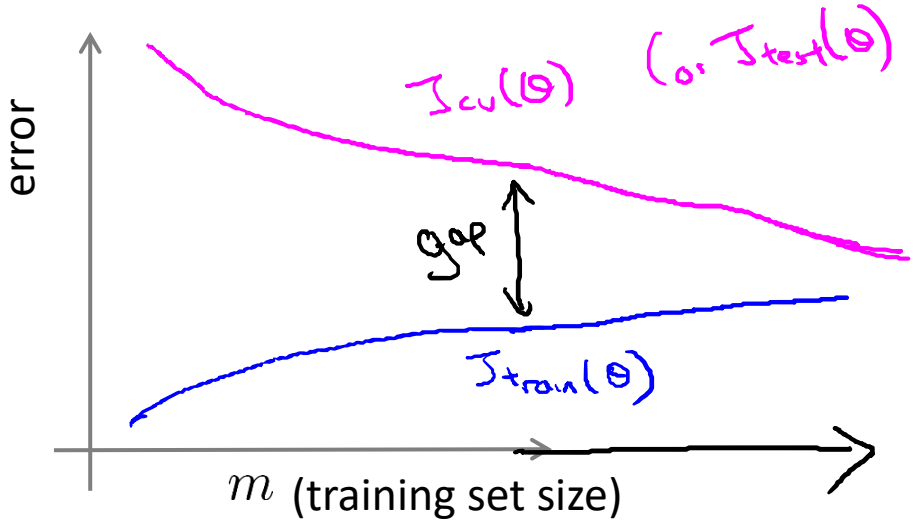

If a learning algorithm is suffering from high variance, getting more training data is likely to help.

$h_\theta(x) = \theta_0 + \theta_1 x + \cdots + \theta_{100} x^{100}$

(and small $\lambda$)



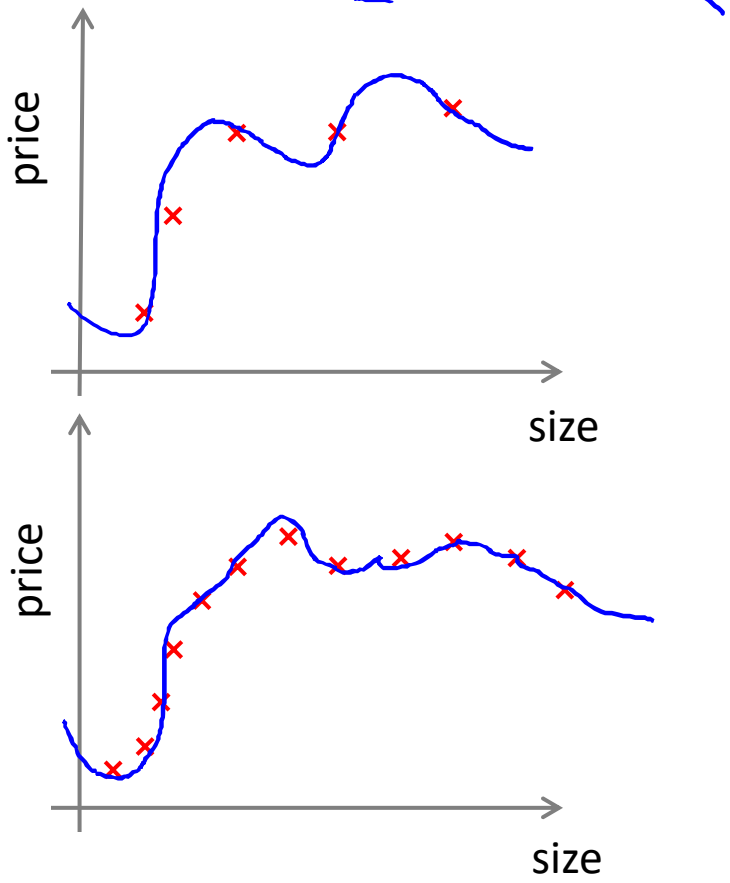
## High variance

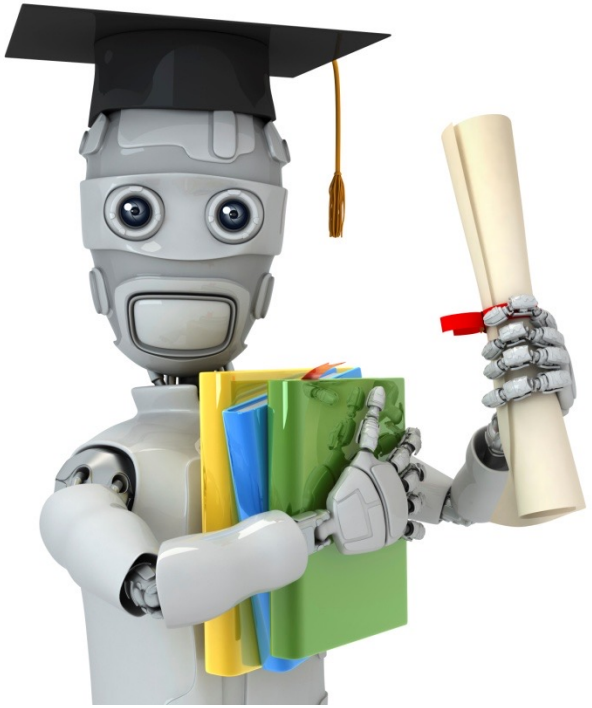



If a learning algorithm is suffering from high variance, getting more training data is likely to help.

$h_\theta(x) = \theta_0 + \theta_1 x + \cdots + \theta_{100} x^{100}$

(and small $\lambda$)





Machine Learning

# Advice for applying machine learning

## Deciding what to try next (revisited)

**Debugging a learning algorithm:**

Suppose you have implemented regularized linear regression to predict housing prices. However, when you test your hypothesis in a new set of houses, you find that it makes unacceptably large errors in its prediction. What should you try next?

- Get more training examples
- Try smaller sets of features
- Try getting additional features
- Try adding polynomial features $(x_1^2, x_2^2, x_1 x_2, \text{etc})$
- Try decreasing $\lambda$
- Try increasing $\lambda$

**Debugging a learning algorithm:**

Suppose you have implemented regularized linear regression to predict housing prices. However, when you test your hypothesis in a new set of houses, you find that it makes unacceptably large errors in its prediction. What should you try next?

- Get more training examples → fixes high variance
- Try smaller sets of features → fixes high variance
- Try getting additional features → fixes high bias
- Try adding polynomial features $(x_1^2, x_2^2, x_1x_2, \text{etc})$ → fixes high bias.
- Try decreasing $\lambda$ → fixes high bias
- Try increasing $\lambda$ → fixes high variance